

Mode-Shape Deformation of Power System DAEs by Time-Domain Integration Methods

Carlo Tajoli, *IEEE Student Member*, Georgios Tzounas, *IEEE Member*, and Gabriela Hug, *IEEE Senior Member*

Power Systems Laboratory

ETH Zürich

Zurich, Switzerland

{tajoli, georgios.tzounas, hug}@eeh.ee.ethz.ch

Abstract—This paper studies the numerical deformation that time-domain integration (TDI) methods introduce to the shape of the coupling between the dynamic modes and variables of power system models. To this aim, we employ a small-signal stability analysis (SSSA)-based framework where such *mode-shape deformation* is efficiently identified by comparing the modal participation factors (PFs) of the power system model with the PFs of the discrete-time system that is derived from the application of the TDI method. The proposed approach is illustrated for TDI methods commonly used in dynamic power system calculations.

Index Terms—time-domain integration (TDI), numerical methods, mode shape, numerical deformation, participation factors (PFs).

I. INTRODUCTION

A. Motivation

The stability analysis of a power system following a large disturbance – such as the sudden loss of an important generator, a line fault, etc. – relies on the solution of a non-linear model of differential-algebraic equations (DAEs) [1]. Power system software tools approximate this solution numerically by running a time-domain simulation routine. However, rapid and precise stability analysis through time-domain simulations is not straightforward, especially with the growing penetration of converter-based resources which significantly increases the dynamic complexity and stiffness of power system models.

B. Literature Review

There exist two time-domain simulation approaches to obtain the solution of the DAEs that describe the dynamics of power systems, namely simultaneous and partitioned [2]. In the simultaneous approach, differential and algebraic equations are solved together as one set at each time step through an implicit integration method, such as the Theta method [3], [4]. In the partitioned approach, on the other hand, differential equations are solved at each step for state variables, whereas algebraic equations are solved separately. The solution of differential equations in this case is typically obtained with an explicit integration method [5]. For example, a family of methods commonly employed in a partitioned-solution setup is that of explicit Adams-Bashforth [6], [7].

G. Tzounas and G. Hug are supported by the Swiss National Science Foundation under project NCCR Automation (grant no. 51NF40 18054); C. Tajoli and G. Hug are supported by project ReMaP.

In contrast to implicit time-domain integration (TDI) methods, explicit methods are known to be prone to numerical instabilities. This limits the ability of these methods to use large integration time steps and has often driven efforts for the development of device models that are numerically robust when combined with a given commercial explicit solver. In this vein, recent works have focused on the formulation and numerical robustness of converter-based resource models for systems with low short-circuit strength, e.g., see [8], [9].

The accuracy of a time-domain simulation is traditionally evaluated through truncation error analysis. Moreover, the numerical stability of a TDI method is conventionally characterized by testing its convergence on a linear scalar equation. Recent studies focused on the development of a framework to assess accuracy and numerical stability of TDI methods in a unified way. In particular, [10]–[12] estimate the numerical distortion that a given TDI method introduces to a power system model by comparing the small-signal dynamic modes of the original model with the modes of the approximated system that results from the application of the method. Such framework allows, first, to extract useful upper time step bounds that satisfy prescribed requirements of precision and over/under-damping; and, second, to provide a fair computational comparison among different methods.

Apart from the numerical error that they cause to the dynamic modes of a power system model, TDI methods may also introduce a spurious deformation to the shape of the coupling between dynamic modes and system variables. This *mode-shape* aspect of numerical deformation has not, to the best of our knowledge, been investigated in the literature. To provide a first study that tackles this aspect is the main goal of our work in this paper.

C. Contributions

The contributions of the paper are twofold, as follows:

- Provision of a small-signal stability analysis (SSSA)-based technique to estimate the numerical mode-shape deformation introduced to DAE power system models by TDI methods.
- Thorough discussion on the mode-shape deformation caused to power system DAEs by well-known TDI methods, including Theta, two-stage diagonally implicit Runge Kutta (2S-DIRK), and Heun’s method (HM).

D. Paper Organization

The remainder of the paper is organized as follows. Section II recalls the formulation and numerical integration of DAE power system models. Section III describes the proposed approach to quantify the numerical deformation of the system's mode shapes caused by TDI methods. Section IV discusses the case study. Conclusions are drawn in Section V.

II. POWER SYSTEM MODEL AND NUMERICAL SOLUTION

A. DAE Model

In short-term stability analysis, the dynamic model of a power system is conventionally formulated as a set of non-linear DAEs, as follows [1]:

$$\begin{aligned} \mathbf{x}'(t) &= \mathbf{f}(\mathbf{x}(t), \mathbf{y}(t)), \\ \mathbf{0}_{\mu,1} &= \mathbf{g}(\mathbf{x}(t), \mathbf{y}(t)). \end{aligned} \quad (1)$$

In (1), $\mathbf{x}(t) : [0, \infty) \rightarrow \mathbb{R}^\nu$ and $\mathbf{y}(t) : [0, \infty) \rightarrow \mathbb{R}^\mu$ are the states and algebraic variables, respectively, of the system; $\mathbf{f} : \mathbb{R}^{\nu+\mu} \rightarrow \mathbb{R}^\nu$ and $\mathbf{g} : \mathbb{R}^{\nu+\mu} \rightarrow \mathbb{R}^\mu$ are non-linear functions; $\mathbf{0}_{\mu,1}$ is the zero matrix of dimensions $\mu \times 1$. For simplicity, discrete system dynamics are not explicitly considered in (1). Readers interested in the modeling and handling of discontinuities are referred to [13] and the bibliography therein.

B. Numerical Integration

The time-domain simulation of a power system model consists in employing a proper numerical method to approximate the solution of (1) for a known set of initial conditions. Every numerical TDI method applied to (1) can be mathematically described as a set of non-linear difference equations whose definition depends on \mathbf{f} and \mathbf{g} . For example, employing the well-known Theta method [3] leads to the following set of difference equations:

$$\begin{aligned} \mathbf{x}_{n+1} &= \mathbf{x}_n + h[\theta \mathbf{f}(\mathbf{x}_n, \mathbf{y}_n) + (1 - \theta) \mathbf{f}(\mathbf{x}_{n+1}, \mathbf{y}_{n+1})], \\ \mathbf{0}_{\mu,1} &= h\mathbf{g}(\mathbf{x}_{n+1}, \mathbf{y}_{n+1}), \end{aligned} \quad (2)$$

where h is the simulation time step size; $0 \leq \theta \leq 0.5$ determines the method's damping; and $\mathbf{x}_{n+1-\ell} : \mathbb{N} \rightarrow \mathbb{R}^\nu$, $\mathbf{y}_{n+1-\ell} : \mathbb{N} \rightarrow \mathbb{R}^\mu$, $\ell = \{0, 1\}$. Given the values of state and algebraic variables at some point $(\mathbf{x}_n, \mathbf{y}_n) := [\mathbf{x}_n^\top, \mathbf{y}_n^\top]^\top$ (where \top is the matrix transpose), the goal at each time step is to compute the new values $(\mathbf{x}_{n+1}, \mathbf{y}_{n+1})$. The latter provides an approximation of the exact solution of (1), i.e:

$$\begin{aligned} \mathbf{x}_{n+1-\ell} &\approx \mathbf{x}(t + (1 - \ell)h), \\ \mathbf{y}_{n+1-\ell} &\approx \mathbf{y}(t + (1 - \ell)h). \end{aligned} \quad (3)$$

The accuracy and convergence of a TDI depends on the time step size h , as well as on the numerical properties of the TDI method employed. For example, (2) for $\theta = 0.5$ corresponds to the trapezoidal method (TM), which always converges for stable and diverges for unstable trajectories. On the other hand, for $\theta = 0$, (2) corresponds to the backward Euler method (BEM), which has very fast convergence but tends to overdamp the dynamics of the system.

C. Model Stiffness and SSSA

System (1) is known to be stiff, i.e. its differential equations span a wide range of time constants [1]. The stiffness of (1) can be measured by the ratio between the largest and smallest eigenvalues of the corresponding small-signal model.

Consider that a stationary solution $(\mathbf{x}_o, \mathbf{y}_o)$ of (1) is known. Then, differentiating (1) at the stationary point gives:

$$\begin{aligned} \tilde{\mathbf{x}}'(t) &= \mathbf{f}_x \tilde{\mathbf{x}}(t) + \mathbf{f}_y \tilde{\mathbf{y}}(t), \\ \mathbf{0}_{\mu,1} &= \mathbf{g}_x \tilde{\mathbf{x}}(t) + \mathbf{g}_y \tilde{\mathbf{y}}(t), \end{aligned} \quad (4)$$

where $\tilde{\mathbf{x}}(t) = \mathbf{x}(t) - \mathbf{x}_o$, $\tilde{\mathbf{y}}(t) = \mathbf{y}(t) - \mathbf{y}_o$; and \mathbf{f}_x , \mathbf{f}_y , \mathbf{g}_x , \mathbf{g}_y are Jacobian matrices evaluated at $(\mathbf{x}_o, \mathbf{y}_o)$. Under the assumption that \mathbf{g}_y is non-singular¹, algebraic variables can be eliminated² and (4) can be rewritten as a set of linear ordinary differential equations, as follows:

$$\tilde{\mathbf{x}}'(t) = \mathbf{A} \tilde{\mathbf{x}}(t), \quad (5)$$

where $\mathbf{A} = \mathbf{f}_x - \mathbf{f}_y \mathbf{g}_y^{-1} \mathbf{g}_x$. Then, stability of (4) is assessed through the eigenvalues of (5), which are obtained from the numerical solution of the algebraic problem [14]:

$$(s\mathbf{I}_\nu - \mathbf{A})\mathbf{v} = \mathbf{0}_{\nu,1}, \quad (6)$$

$$\mathbf{w}(s\mathbf{I}_\nu - \mathbf{A}) = \mathbf{0}_{1,\nu}, \quad (7)$$

where s denotes a complex frequency in the S -domain; \mathbf{I}_ν denotes the identity matrix of dimensions $\nu \times \nu$; $\mathbf{v} \in \mathbb{C}^{\nu \times 1}$ and $\mathbf{w} \in \mathbb{C}^{1 \times \nu}$. Every s_i , $i = \{1, 2, \dots, \nu\}$, that satisfies (6) is an eigenvalue of \mathbf{A} , with \mathbf{v}_i , \mathbf{w}_i being the corresponding right and left, respectively, eigenvectors. Then, the system is asymptotically stable if $\forall s_i$, $\Re(s_i) < 0$. Let the system be stable and s^{\max} , s^{\min} be the eigenvalues with largest and smallest magnitudes, i.e. $s^{\max} = \max |s_i|$, $s^{\min} = \min |s_i|$, $\forall s_i$, then the stiffness ratio of (1) can be defined as follows:

$$\mathcal{S} = |s^{\max}| / |s^{\min}|. \quad (8)$$

III. PROPOSED APPROACH

A. SSSA of Integration Methods

The small-disturbance properties of a TDI method applied to a power system model can be seen by studying a linear system of difference equations in the form [10], [12]:

$$\mathbf{y}_{n+1} = \mathbf{G} \mathbf{y}_n, \quad (9)$$

where $\mathbf{y}_n : \mathbb{N} \rightarrow \mathbb{R}^q$. Equation (9) is a discrete-time approximation of (5), where \mathbf{G} varies for different TDI methods but is always a function of \mathbf{A} and h . For the sake of example, consider the Theta method described by (2). Differentiating (2) at $(\mathbf{x}_o, \mathbf{y}_o)$ gives:

$$\begin{aligned} \tilde{\mathbf{x}}_{n+1} &= \tilde{\mathbf{x}}_n + h[\theta(\mathbf{f}_x \tilde{\mathbf{x}}_n + \mathbf{f}_y \tilde{\mathbf{y}}_n) \\ &\quad + (1 - \theta)(\mathbf{f}_x \tilde{\mathbf{x}}_{n+1} + \mathbf{f}_y \tilde{\mathbf{y}}_{n+1})], \end{aligned} \quad (10)$$

$$\mathbf{0}_{\mu,1} = \mathbf{g}_x \tilde{\mathbf{x}}_{n+1} + \mathbf{g}_y \tilde{\mathbf{y}}_{n+1}. \quad (11)$$

¹In this paper, we assume that \mathbf{g}_y is invertible. This assumption comes with no loss of generality, as potential singularities of \mathbf{g}_y can be always eliminated by reformulating (1) to an equivalent DAE set with non-singular \mathbf{g}_y .

²Eliminating $\tilde{\mathbf{y}}$ is the best approach for small/medium size systems. In large systems it is more efficient to maintain sparsity and work directly with (4).

From (11) we have that $\tilde{\mathbf{y}}_{n+1} = -\mathbf{g}_y^{-1}\mathbf{g}_x\tilde{\mathbf{x}}_{n+1}$ and $\tilde{\mathbf{y}}_n = -\mathbf{g}_y^{-1}\mathbf{g}_x\tilde{\mathbf{x}}_n$, and (10)-(11) can be rewritten as follows:

$$[\mathbf{I}_\nu - h(1-\theta)\mathbf{A}]\tilde{\mathbf{x}}_{n+1} = (\mathbf{I}_\nu + h\theta\mathbf{A})\tilde{\mathbf{x}}_n, \quad (12)$$

or equivalently,

$$\tilde{\mathbf{x}}_{n+1} = [\mathbf{I}_\nu - h(1-\theta)\mathbf{A}]^{-1}(\mathbf{I}_\nu + h\theta\mathbf{A})\tilde{\mathbf{x}}_n, \quad (13)$$

which is a system in the form of (9), where $\mathbf{y}_n \equiv \tilde{\mathbf{x}}_n$, and:

$$\mathbf{G} = [\mathbf{I}_\nu - h(1-\theta)\mathbf{A}]^{-1}(\mathbf{I}_\nu + h\theta\mathbf{A}). \quad (14)$$

The eigenvalue problem associated to (9) is:

$$(\hat{z}\mathbf{I}_\nu - \mathbf{G})\mathbf{v} = \mathbf{0}_{q,1}, \quad (15)$$

$$\mathbf{w}(\hat{z}\mathbf{I}_\nu - \mathbf{G}) = \mathbf{0}_{1,q}, \quad (16)$$

where \hat{z} is a complex frequency in the Z -domain. Then, (9) is asymptotically stable if and only if $|\hat{z}_j| < 1 \forall \hat{z}_j$, $j = 1, 2, \dots, q$ that satisfies (15), (16). Comparison of the eigenvalues of \mathbf{G} and \mathbf{A} provides a rough yet accurate estimate of the numerical deformation that a given TDI method introduces when applied to (1) [10], [12]. Obviously, for the eigenvalues of the two matrices to be comparable, they need to be referred to the same plane through the map $z = e^{sh}$. Let s_i be an eigenvalue of \mathbf{A} and \hat{z}_j be the corresponding eigenvalue as deformed by the TDI method. Then, the associated numerical deformation can be estimated through the relative error:

$$\epsilon_s = 100(|s_i - \log(\hat{z}_j)/h|/|s_i|). \quad (17)$$

B. Deformation of Mode Shapes

Apart from the numerical deformation that they introduce to the dynamic modes of a model, TDI methods may also deform the coupling shape of dynamic modes and state variables. In this section, we describe the proposed approach to estimate such mode-shape deformation.

In the context of SSSA, the information of mode shapes for a given system is included in its right and left eigenvectors [14]. Given the eigenvectors of a system, an efficient measure of the shape of coupling between states and variables is provided through modal participation analysis [15]. Consider system (5): If s_i is an eigenvalue of \mathbf{A} and \mathbf{v}_i , \mathbf{w}_i are the associated eigenvectors, then the corresponding modal participation factor (PF) is defined as the dimensionless number:³

$$p = w_{i,k} v_{k,i}, \quad (18)$$

where $v_{k,i}$ is the k -th row element of \mathbf{v}_i and $w_{i,k}$ is the k -th column element of \mathbf{w}_i . The PF in (18) represents the relative contribution of the i -th mode s_i in the response of the k -th state variable x_k . Note that PFs can be collected to form the system's participation matrix \mathbf{P} , as follows:

$$\mathbf{P} = \mathbf{W}^\top \circ \mathbf{U}, \quad (19)$$

where \circ denotes component-wise matrix multiplication; \mathbf{U} is the modal matrix with the right eigenvectors as columns, and

³Definition (18) assumes that the algebraic multiplicities of all eigenvalues equal the geometric ones. The reader interested in modal participation analysis of systems that do not satisfy this assumption is referred to [16].

\mathbf{W} is the modal matrix with the left eigenvectors as rows, i.e. $\mathbf{U} = [\mathbf{v}_1 \ \mathbf{v}_2 \ \dots \ \mathbf{v}_\nu]$, $\mathbf{W} = [\mathbf{w}_1^\top \ \mathbf{w}_2^\top \ \dots \ \mathbf{w}_\nu^\top]^\top$.

Now, consider a TDI method and the associated approximated system (9). The modal participation matrix associated to (9) is then defined as follows [14]:

$$\mathbf{\Pi} = \mathbf{W}^\top \circ \mathbf{V}, \quad (20)$$

with $\mathbf{V} = [\mathbf{v}_1 \ \mathbf{v}_2 \ \dots \ \mathbf{v}_q]$, $\mathbf{W} = [\mathbf{w}_1^\top \ \mathbf{w}_2^\top \ \dots \ \mathbf{w}_q^\top]^\top$. Note that matrix $\mathbf{\Pi}$ basically represents an approximation of the participation matrix \mathbf{P} . If p is an element of \mathbf{P} and π is the corresponding element of $\mathbf{\Pi}$, then the quantity:

$$\epsilon_p = 100(|\pi| - |p|)/|p|, \quad (21)$$

provides an estimate of the associated relative mode-shape deformation introduced by the TDI method.

We note that metrics (17) and (21) are based on SSSA and thus they are technically valid around stationary solutions. Yet, the structure and stiffness of (1) as well as the properties of TDI methods are features that tend to be "robust" and hence results provide also a tentative yet accurate estimate of deformation also for varying operating conditions. For similar considerations we refer to the literature, e.g., [10], [17], [18].

C. Deformation by Common Methods

In this section, we discuss the mode-shape deformation introduced by well-known TDI methods used for the simulation of power system dynamics. We first show that certain methods do not deform at all the mode-shapes of dynamic modes that are represented by *non-degenerate* eigenvalues, i.e. eigenvalues with algebraic multiplicity equal to 1. To this aim, we provide the following result from linear algebra.

Consider two commuting matrices \mathbf{A} and \mathbf{G} :

$$\mathbf{A}\mathbf{G} = \mathbf{G}\mathbf{A}. \quad (22)$$

If \mathbf{v}_i is a right eigenvector of \mathbf{A} corresponding to the non-degenerate eigenvalue s_i , then it is also an eigenvector of \mathbf{G} .

Proof. The eigenvalue problem associated to \mathbf{A} is (6), whereby substituting \mathbf{v}_i , s_i and pre-multiplying by \mathbf{G} we get:

$$\begin{aligned} (s_i\mathbf{G} - \mathbf{G}\mathbf{A})\mathbf{v}_i &= (s_i\mathbf{G} - \mathbf{A}\mathbf{G})\mathbf{v}_i \\ &= (s_i\mathbf{I}_\nu - \mathbf{A})\mathbf{G}\mathbf{v}_i = \mathbf{0}_{\nu,1}. \end{aligned} \quad (23)$$

Thus, $\mathbf{G}\mathbf{v}_i$ is also a right eigenvector of \mathbf{A} associated to s_i or, equivalently, $\mathbf{G}\mathbf{v}_i$ is proportional to \mathbf{v}_i :

$$(\lambda\mathbf{I}_\nu - \mathbf{G})\mathbf{v}_i = \mathbf{0}_{\nu,1}, \quad (24)$$

i.e., \mathbf{v}_i is an eigenvector of \mathbf{G} associated to the eigenvalue λ . Note that if \mathbf{G} represents a TDI method, as is the case in this work, then $\lambda \equiv \hat{z}_i$. The reciprocal case of left eigenvectors can be derived similarly and thus, for a non-degenerate eigenvalue and a method whose matrix \mathbf{G} commutes with \mathbf{A} , we have that $|p| = |\pi|$ in (21), or, $\epsilon_p = 0$.

1) *Theta method*: Consider the Theta method (2), for which \mathbf{G} is given by (14). To prove commutativity of \mathbf{G} and \mathbf{A} , we start by considering the identity $\mathbf{A} - c\mathbf{A}^2 = \mathbf{A} - c\mathbf{A}^2$, which can be equivalently rewritten as:

$$(\mathbf{I}_\nu - c\mathbf{A})\mathbf{A} = \mathbf{A}(\mathbf{I}_\nu - c\mathbf{A}). \quad (25)$$

Left and right multiplication by $(\mathbf{I}_\nu - c\mathbf{A})^{-1}$ yields:

$$\mathbf{A}(\mathbf{I}_\nu - c\mathbf{A})^{-1} = (\mathbf{I}_\nu - c\mathbf{A})^{-1}\mathbf{A}. \quad (26)$$

Right multiplication of both sides of (26) by $b\mathbf{A}$, $b \in \mathbb{R}$ gives:

$$\mathbf{A}(\mathbf{I}_\nu - c\mathbf{A})^{-1}b\mathbf{A} = (\mathbf{I}_\nu - c\mathbf{A})^{-1}b\mathbf{A}^2. \quad (27)$$

Summing (26) and (27) and using $c = h(1 - \theta)$, $b = h\theta$, leads to:

$$\begin{aligned} \mathbf{A}[\mathbf{I}_\nu - h(1 - \theta)\mathbf{A}]^{-1}(\mathbf{I}_\nu + h\theta\mathbf{A}) &= \\ &= [\mathbf{I}_\nu - h(1 - \theta)\mathbf{A}]^{-1}(\mathbf{I}_\nu + h\theta\mathbf{A})\mathbf{A}, \end{aligned} \quad (28)$$

or, equivalently, to (22). The proof is complete.

2) *BEM and TM*: They are special cases of the Theta method. For $\theta = 0$, BEM is obtained and (28) becomes:

$$\mathbf{A}(\mathbf{I}_\nu - h\mathbf{A})^{-1} = (\mathbf{I}_\nu - h\mathbf{A})^{-1}\mathbf{A}. \quad (29)$$

TM is obtained for $\theta = 0.5$, in which case (28) becomes:

$$\begin{aligned} \mathbf{A}(\mathbf{I}_\nu - 0.5h\mathbf{A})^{-1}(\mathbf{I}_\nu + 0.5\theta\mathbf{A}) &= \\ &= (\mathbf{I}_\nu - 0.5h\mathbf{A})^{-1}(\mathbf{I}_\nu + 0.5\theta\mathbf{A})\mathbf{A}. \end{aligned} \quad (30)$$

3) *2S-DIRK*: We consider the 2S-DIRK proposed in [19] for the simulation of electromagnetic transients. The method's first stage computes the solution at an intermediate point:

$$\begin{aligned} \chi_{n+1} &= \mathbf{x}_n + \alpha h \mathbf{f}(\chi_{n+1}, \psi_{n+1}), \\ \mathbf{0}_{\mu,1} &= h\mathbf{g}(\chi_{n+1}, \psi_{n+1}), \end{aligned} \quad (31)$$

where $\alpha = 1 - 1/\sqrt{2}$. Then, χ_{n+1} is used to calculate:

$$\chi_n = \beta \mathbf{x}_n + (1 - \beta) \chi_{n+1}, \quad \beta = -\sqrt{2}. \quad (32)$$

The final solution is obtained from the following equations:

$$\begin{aligned} \mathbf{x}_{n+1} &= \chi_n + \alpha h \mathbf{f}(\mathbf{x}_{n+1}, \mathbf{y}_{n+1}), \\ \mathbf{0}_{\mu,1} &= h\mathbf{g}(\mathbf{x}_{n+1}, \mathbf{y}_{n+1}). \end{aligned} \quad (33)$$

Differentiating (31)-(33) at $(\mathbf{x}_o, \mathbf{y}_o)$ allows expressing the method in the form of (9), where [10]:

$$\mathbf{G} = (\mathbf{I}_\nu - \alpha h\mathbf{A})^{-1}(\mathbf{I}_\nu - \alpha\beta h\mathbf{A})(\mathbf{I}_\nu - \alpha h\mathbf{A})^{-1}. \quad (34)$$

Matrices \mathbf{A} , \mathbf{G} in (34) are commuting. We omit the proof due to space constraints, but it can be easily constructed similarly to the Theta method starting from (26), where in this case $c = \alpha h$.

Methods 1)-3) above are implicit methods commonly used in a simultaneous-solution approach setup. We have shown that these methods do not deform the mode shape of dynamics represented by non-degenerate eigenvalues. This is an important result since critical modes that dominate the dynamic response of power system models are typically represented by non-degenerate eigenvalues. The deformation introduced by these TDI methods is further discussed through simulations in the case study presented in Section IV.

4) *Heun's Method (HM)*: We consider an element of the family of explicit Adams-Bashforth methods, namely HM. Variants of HM are commonly employed by software tools that adopt the partitioned-solution approach. In HM, a predictor provides an initial estimate ($\xi_{n+1}^{(0)}$) of \mathbf{x}_{n+1} , as follows:

$$\xi_{n+1}^{(0)} = \mathbf{x}_n + h\mathbf{f}(\mathbf{x}_n, \mathbf{y}_n). \quad (35)$$

Then, accuracy of the current estimation is refined through corrector steps. The i -th corrector step has the form:

$$\xi_{n+1}^{(i)} = \mathbf{x}_n + 0.5h\mathbf{f}(\mathbf{x}_n, \mathbf{y}_n) + 0.5h\mathbf{f}(\xi_{n+1}^{(i-1)}, \mathbf{y}_n), \quad (36)$$

with $i \in \mathbb{N}^* : i \leq r$, where typically $r = 1$ or 2. HM needs to be combined with a way to deal with interfacing of algebraic variables [2]. In (36), such interfacing is achieved by extrapolation, i.e. \mathbf{y}_n is used instead of \mathbf{y}_{n+1} in the last term of the right-hand side of (36) [2], [12]. Then, $(\mathbf{x}_{n+1}, \mathbf{y}_{n+1})$ is obtained from:

$$\mathbf{x}_{n+1} = \xi_{n+1}^{(r)}, \quad (37)$$

$$\mathbf{0} = h\mathbf{g}(\mathbf{x}_{n+1}, \mathbf{y}_{n+1}). \quad (38)$$

Differentiation of (36)-(38) at $(\mathbf{x}_o, \mathbf{y}_o)$ allows expressing the method in the form of (9), where:

$$\mathbf{G} = \mathbf{I}_\nu + h \sum_{j=0}^r \left(\frac{h}{2} \mathbf{f}_x \right)^j \mathbf{A}, \quad (39)$$

with $r \in \mathbb{N}^*$. The proof of (39) can be found in [12]. For $r \geq 1$, \mathbf{A} and \mathbf{G} in (39) do not commute, which implies that HM is expected to deform the mode shapes of both degenerate and non-degenerate eigenvalues. If $r = 0$, HM reduces to the forward Euler method (FEM) and (39) yields $\mathbf{G} = \mathbf{I}_\nu + h\mathbf{A}$. In this case $\mathbf{G}\mathbf{A} = \mathbf{A}\mathbf{G} = \mathbf{A} + h\mathbf{A}^2$. Yet, FEM is known to show a poor performance, which from the viewpoint of this paper implies that the method gives rise to very large ϵ_s errors. Thus, in the remainder of this work, FEM is not considered.

IV. CASE STUDY

In this section, we illustrate the proposed approach through simulations carried out based on the IEEE 39-bus test system. The IEEE 39-bus system includes 10 synchronous generators (SGs) represented by a 4-th order model, 34 lines, 12 transformers, and 19 loads. All SGs are equipped with primary frequency and voltage regulators, and power system stabilizers. The system's static and dynamic data can be found in [20]. Simulations in this section are carried out using Dome [21].

A. Eigenvalue Deformation

The eigenvalues of the DAE system obtained from (6) are compared to the ones of the associated problem (15) for Theta, 2S-DIRK, and HM; numerical deformation is calculated as in (17) for different time step sizes h . Figure 1 shows the spurious shift that these methods introduce to the rightmost eigenvalues of the system. While HM already presents considerable deviations from the exact system dynamics for $h = 0.01$ s, Theta and 2S-DIRK have a good performance and notably deteriorate only for h in the order of 10^{-1} s or higher.

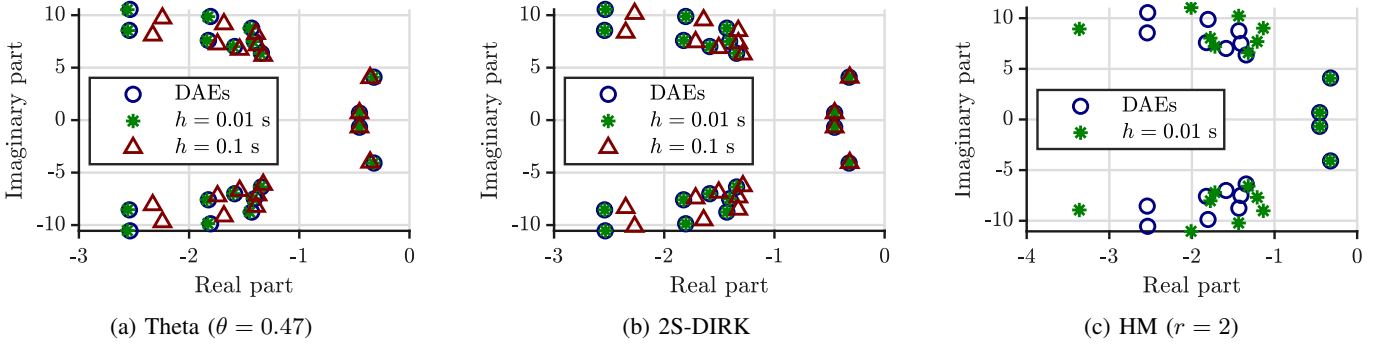


Fig. 1: Eigenvalue deformation for Theta, 2S-DIRK and HM.

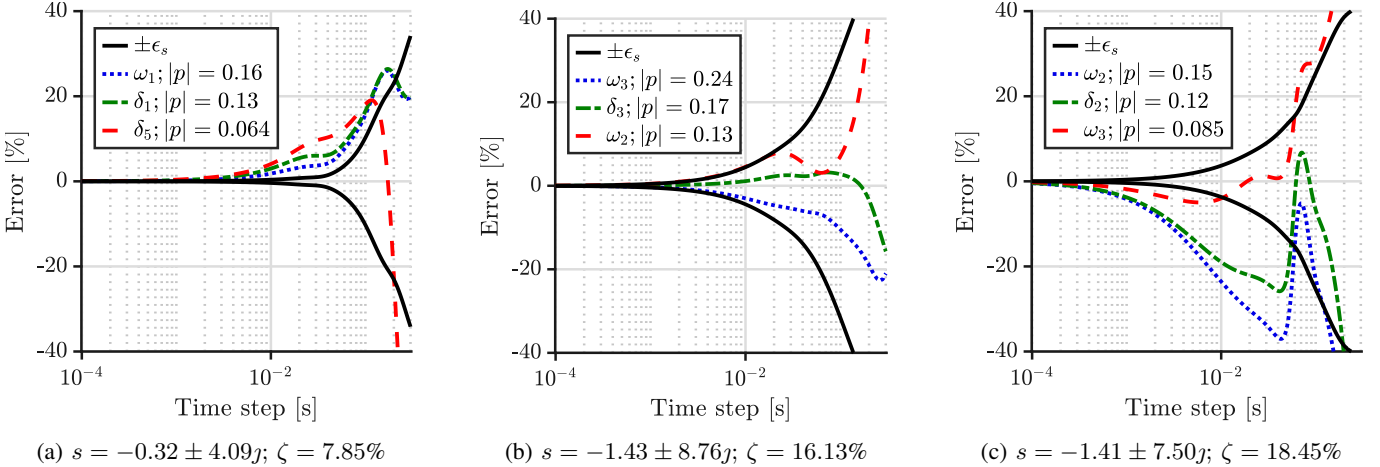


Fig. 2: PF deformation ϵ_p as a function of h for 3 poorly damped modes (with ζ denoting the damping ratio). In each plot, the states with the highest PFs are represented, along with the associated eigenvalue deformation ϵ_s .

B. Deformation of Mode Shapes

We focus on the deformation that TDI methods introduce to the coupling between system states and variables. To this aim, the participation matrices \mathbf{P} and $\mathbf{\Pi}$ are calculated from (19) and (20), respectively. Since different columns of \mathbf{P} , $\mathbf{\Pi}$ refer to different modes, the columns of $\mathbf{\Pi}$ are sorted to pair correctly to the modes of the DAE system. Moreover, the columns of both matrices are normalized so that for every eigenvalue the sum of all PFs is equal to 1. Then, mode-shape deformation for each TDI method is estimated from (21). For Theta and 2S-DIRK, we find that $\epsilon_p = 0$ for all non-degenerate eigenvalues, which is consistent with the discussion of Section III-C. Furthermore, for degenerate eigenvalues, large values of ϵ_p are observed in some cases. Yet, these cases are always associated with very low PFs ($|p| < 10^{-3}$). Given that the behavior of a variable is largely defined by a small number of highly participating modes (often by 1 or 2), the numerical impact of these cases on the system is negligible.

For HM, significant values of ϵ_p are observed for both non-degenerate and degenerate eigenvalues. Figure 2 shows, for the most poorly damped electromechanical modes, how ϵ_p varies as a function of h when 2 corrector steps are used. For the sake of comparison, the corresponding ϵ_s is included in each plot. In Fig. 2, δ_i , ω_i denote the rotor angle and speed, respectively,

of the i -th SG. The deformation presents an irregular behavior but generally increases with the step size. Very small steps lead to good accuracy but also lead to a high computational burden. Note also that for several modes and step sizes the maximum ϵ_p is higher than ϵ_s . For the most critical mode, for example (Fig. 2a), $h = 0.07$ s leads to $\epsilon_s < 5\%$ but also to a maximum ϵ_p of $\epsilon_p^{\max} > 9\%$. Assuming for this mode a prescribed accuracy degree of $\epsilon_s, \epsilon_p^{\max} < 5\%$, the maximum admissible time step is estimated at $h^{\max} = 0.012 < 0.07$ s. Another example is shown in Fig. 2c, where $h^{\max} = 1$ ms is needed to maintain $\epsilon_p^{\max} < 5\%$, although ϵ_s is low even for $h = 0.01$ s. The above discussion highlights the relevance of evaluating both metrics in a numerical analysis.

Table I reports, for different methods, the maximum admissible time step h^{\max} in 4 scenarios: (i) $\epsilon_s < 5\%$, (ii) $\epsilon_p < 5\%$, (iii) $\epsilon_p < 10\%$, and (iv) $\epsilon_s, \epsilon_p < 5\%$. In all cases, h^{\max} is obtained considering the 5 most critical eigenvalues – which are all non-degenerate – and for each eigenvalue, the 3 largest PFs. Results show that h^{\max} for implicit methods is about an order of magnitude larger than HM and is not impacted by mode-shape deformation. On the other hand, h^{\max} for HM is largely impacted by the selected ϵ_p threshold. We note that in practice $\epsilon_p^{\max}, \epsilon_s^{\max}$ can be setup for any prescribed requirements. In this regard, a relevant question

TABLE I: h^{\max} estimated for the 5 most critical modes and 3 highest PFs. Theta and 2S-DIRK have $\epsilon_p = 0$.

Accuracy	Method			
	Theta	2S-DIRK	HM ($r = 1$)	HM ($r = 2$)
$\epsilon_s < 5\%$	0.080	0.115	0.0087	0.0098
$\epsilon_p < 5\%$	∞	∞	0.0012	0.0012
$\epsilon_p < 10\%$	∞	∞	0.0026	0.0027
$\epsilon_s, \epsilon_p < 5\%$	0.080	0.115	0.0012	0.0012

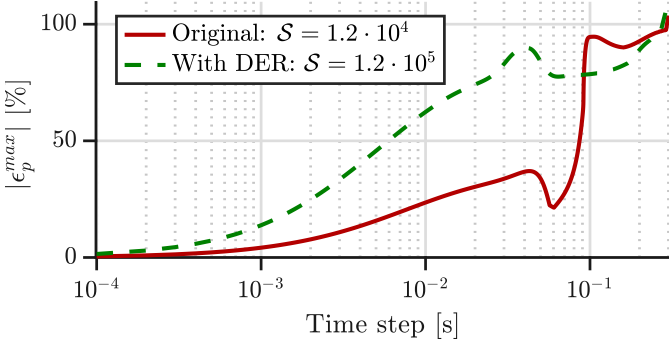


Fig. 3: Effect of stiffness on HM mode-shape deformation. The curves show ϵ_p^{\max} for the original and modified systems.

that is worth further study is how to best tune ϵ_p^{\max} , ϵ_s^{\max} . A good starting point in this direction can be the analytical solution of the linearized system, which depends linearly on eigenvectors (and thus on mode shapes), but exponentially on system eigenvalues, which directs that $\epsilon_p^{\max} > \epsilon_s^{\max}$.

C. Modified System with DERs and AGC

In this section, the test system is modified as follows. SGs at buses 32, 33, 34 and 35 are replaced by aggregated, converter-based distributed energy resource (DER) models. Each DER synchronizes to the grid through a synchronous reference frame phase-locked loop (PLL) and provides primary frequency and voltage support by regulating, at the point of connection, the d and q axis current components, respectively, in the dq reference frame. Moreover, SGs are assumed to provide secondary frequency support through an automatic generation control (AGC) scheme modeled as an integral regulator. The DER PLL dynamics are faster than the fastest dynamics of the original system, while the AGC dynamics are slower than the slowest dynamics of the original system. As a consequence, the system's stiffness ratio, defined as in (8), increases by an order of magnitude (from $1.2 \cdot 10^4$ to $1.2 \cdot 10^5$). Figure 3 shows, as a function of h , the relative error ϵ_p^{\max} for the 3 most participating states of each of the 5 least damped modes. It is seen that increasing the system's stiffness results in a higher distortion, particularly for small time steps.

V. CONCLUSION

This paper studies the numerical deformation that TDI methods cause to the mode shapes of power system DAE models. It is shown that, owing to matrix commutativity properties, common implicit methods, such as Theta and 2S-DIRK,

do not deform the mode shape of dynamics represented by non-degenerate eigenvalues. Moreover, the well-known HM is employed to illustrate through simulations the effect on mode shapes for the case that commutativity properties do not hold. In future work, we will employ the proposed approach to study the numerical robustness of state-of-art converter-based models integrated with explicit TDI methods.

REFERENCES

- [1] P. Kundur, *Power System Stability and Control*. New York: Mc-Grall Hill, 1994.
- [2] J. Machowski, Z. Lubosny, J. W. Bialek, and J. R. Bumby, *Power system dynamics: stability and control*. John Wiley & Sons, 2020.
- [3] J. Sanchez-Gasca, R. D'Aquila, W. Price, and J. Paserba, "Variable time step, implicit integration for extended-term power system dynamic simulation," in *Proceedings of Power Industry Computer Applications Conference*, 1995, pp. 183–189.
- [4] DiGSILENT Power System Solutions, "DiGSILENT PowerFactory," digsilent.de/powerfactory.
- [5] B. Stott, "Power system dynamic response calculations," *Proceedings of the IEEE*, vol. 67, no. 2, pp. 219–241, Feb. 1979.
- [6] PSS/E 33.0, *Program Application Guide Volume 2*. Siemens, 2011.
- [7] General Electric Energy Consulting, "General Electric (GE) PSLF," geenergyconsulting.com/practice-area/software-products/pslf.
- [8] D. Ramasubramanian, W. Wang, P. Pourbeik, E. Farantatos, A. Gaikwad, S. Soni, and V. Chadliev, "Positive sequence voltage source converter mathematical model for use in low short circuit systems," *IET Generation, Transmission & Distribution*, vol. 14, no. 1, pp. 87–97, 2020.
- [9] D. Ramasubramanian, X. Wang, S. Goyal, M. Dewadasa, Y. Li, R. O'Keefe, and P. Mayer, "Parameterization of generic positive sequence models to represent behavior of inverter based resources in low short circuit scenarios," *Electric Power Systems Research*, vol. 213, p. 108616, 2022.
- [10] G. Tzounas, I. Dassios, and F. Milano, "Small-signal stability analysis of numerical integration methods," *IEEE Transactions on Power Systems*, vol. 37, no. 6, pp. 4796–4806, Nov. 2022.
- [11] —, "Small-signal stability analysis of implicit integration methods for power systems with time delays," *Electric Power System Research*, vol. 211, no. 108266, Oct. 2022.
- [12] G. Tzounas and G. Hug, "Unified numerical stability and accuracy analysis of the partitioned-solution approach," *submitted to IEEE Transactions on Power Systems*, 2022, under review. Available at: n.ethz.ch/~gtzounas/pap/psastab.pdf.
- [13] F. Milano, M. Liu, M. A. A. Murad, G. M. Jónsdóttir, G. Tzounas, M. Adeen, Á. Ortega, and I. Dassios, "Power system modelling as stochastic functional hybrid differential-algebraic equations," *IET Smart Grid*, vol. 5, no. 5, pp. 309–331, Oct. 2022.
- [14] F. Milano, I. Dassios, M. Liu, and G. Tzounas, *Eigenvalue Problems in Power Systems*. CRC Press, Taylor & Francis Group, 2020.
- [15] I. J. Pérez-Arriaga, G. C. Verghese, and F. C. Schweppe, "Selective modal analysis with applications to electric power systems, part i: Heuristic introduction," *IEEE Transactions on Power Apparatus and Systems*, vol. PAS-101, no. 9, pp. 3117–3125, Sep. 1982.
- [16] I. Dassios, G. Tzounas, and F. Milano, "Participation factors for singular systems of differential equations," *Circuits, Systems, and Signal Processing*, vol. 39, no. 1, pp. 83–110, 2020.
- [17] G. C. Verghese, I. J. Pérez-Arriaga, and F. C. Schweppe, "Selective modal analysis with applications to electric power systems, part ii: the dynamic stability problem," *IEEE Transactions on Power Apparatus and Systems*, vol. PAS-101, no. 9, pp. 3126–3134, Sep. 1982.
- [18] J. H. Chow, *Power System Coherency and Model Reduction*, ser. Power Electronics and Power Systems 94. New York: Springer-Verlag, 2013.
- [19] T. Noda, K. Takenaka, and T. Inoue, "Numerical integration by the 2-stage diagonally implicit Runge-Kutta method for electromagnetic transient simulations," *IEEE Transactions on Power Delivery*, vol. 24, no. 1, pp. 390–399, 2009.
- [20] Illinois Center for a Smarter Electric Grid (ICSEG), "IEEE 39-Bus System," publish.illinois.edu/smartergrid/ieee-39-bus-system/.
- [21] F. Milano, "A Python-based software tool for power system analysis," in *Proceedings of the IEEE PES General Meeting*, Jul. 2013.