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Robust stability criterion for perturbed singular systems of linearized differential equations

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ABSTRACT

In this article, we consider a class of singular linear systems of differential equations whose coefficients are constant matrices, and study the response of its stability after a perturbation is applied into the system. We use a linear fractional transformation and through its properties we provide a practical test for robust stability. This test requires only the knowledge of the invariants of the initial system. This means it can be used without resorting to any further processes of computations to obtain invariants of any other perturbed system. Finally, numerical examples are given to support and discuss practical applications of the proposed theory.

1. Introduction

Singular systems of linear differential/difference equations are inherent in many physical, engineering, mechanical, and financial models. For instance, in finance, we cite the well-known input–output Leontief model and its several important extensions, see [1,2]. Another application of a singular system is the constrained mechanical and robotic system described in [3]. Singular systems also appear in control theory, see [4], in macroeconomics, see [5], circuit theory, see [6], and in the modeling of power systems, see [7–9].

We consider the following system:

$$EY'(t) = AY(t). \tag{1}$$

where $E, A \in \mathbb{R}^{r\times m}$, $Y : [0, +\infty) \to \mathbb{R}^{m\times 1}$. The matrices E, A can be non-square ($r \neq m$) or square ($r = m$) with E singular ($\det E = 0$). With Y' we denote the first order derivative of $Y(t)$. By applying the Laplace transform $\mathcal{L}$ into (1), we get:

$$E\mathcal{L}\{Y'(t)\} = A\mathcal{L}\{Y(t)\}\ ,$$

or, equivalently,

$$E(sZ(s) - Y_0) = AZ(s)\ .$$

where $s \in \mathbb{C}$, $Y_0 = Y(0)$ initial condition of (1). If we assume that Y_0 is unknown we can use an unknown constant vector $C \in \mathbb{R}^{m\times 1}$ and give to the above expression the following form:

$$(sE - A)Z(s) = EC\ .$$

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From the above equation, it is obvious that the polynomial matrix $sE-A$ plays an important role in the study of (1), especially regarding the existence of solutions and its stability properties.

We will refer to $sE-A$ as the pencil of system (1). Hence, a matrix pencil is a family of matrices $sE-A$, parametrized by a complex number s, see chapter 12 in [10].

Notice that we have two cases:

(a) The first case is $r=m$ and $\det(sE-A)$ to be equal to a polynomial with order less than m, i.e. $\det(sE-A)$ is not identically zero; In this case the pencil $sE-A$ is called *regular pencil*.
(b) The second case is $r\neq m$, or $r=m$ with $\det(sE-A)\equiv 0$, $\forall$ arbitrary $s\in\mathbb{C}$; In this case the pencil $sE-A$ is called *singular pencil*.

In the case of (a), since the pencil is assumed regular, we have that $\det(sE-A)\not\equiv 0$. Then $Z(s)$ can be defined and consequently $Y(t)$ always exists and is given by $Y(t)=\mathcal{L}^{-1}\{(sE-A)^{-1}EC\}$. Hence in the case of a regular pencil, the solution of (1) always exists. In the case of (b), if $r<m$ there are at least $m-r$ unknown functions and m equations. Hence $Z(s)$ cannot be defined uniquely. In this article, we consider the case that the pencil $sE-A$ is *regular* with E *singular*. This type of pencil, see [11], chapter 12 in [10,12], has finite eigenvalues which are the zeros of the function $\det(sE-A)$, and eigenvalues s that tend to infinity. The existence of an infinite eigenvalue in pencils of singular systems can be seen if we write the generalized eigenvalue problem in the reciprocal form $EX=s^{-1}AX$. If E is singular with a null vector X, then $EX=0_{m,1}$, so that X is an eigenvector of the reciprocal problem corresponding to the eigenvalue $s^{-1}=0$, i.e. $s\to\infty$.

We consider now the pencil $sE-A$ of system (1). If we replace s into the characteristic equation $\left|sE-A\right|=0$ with

$$s:=f(z)=\frac{az+b}{cz+d},\quad a,b,c,d\in\mathbb{C},\quad ad-bc\neq 0,\tag{2}$$

we get

$$\left|\frac{az+b}{cz+d}E-A\right|=0,$$

or, equivalently, by using determinant properties

$$\left|(aE-cA)z-(dA-bE)\right|=0,$$

which is the characteristic equation of a linear dynamical system

$$(aE-cA)\tilde{Y}'(t)=(dA-bE)\tilde{Y}(t),$$

with pencil

$$z(aE-cA)-(dA-bE).$$

The transformation (2) used is called Möbius transformation, or linear fractional transformation, and the restriction in this definition is necessary because if $ad=bc$ then s is constant, which would disqualify such transformation here, see [13].

If we consider the transformation (2) for $a=c=d=1$, and $b=-1$, we get

$$F\tilde{Y}'=G\tilde{Y},\tag{3}$$

with pencil

$$Fz-G,$$

where we set

$$F=E-A,\quad G=A+E.$$

The equilibrium state in system (1) is asymptotically stable if the finite eigenvalues of $sE-A$ belong in the set $\{Re(w)<0,\quad\forall w\in\mathbb{C}\}$. By applying the transformation (2) for $a=c=d=1$, and $b=-1$ to this set we get:

$$\frac{z-1}{z+1}+\frac{\bar{z}-1}{\bar{z}+1}<0,$$

or, equivalently,

$$(\bar{z}+1)(z-1)+(\bar{z}-1)(z+1)<0,$$

or, equivalently, by taking into account that $\bar{z}z=|z|^2$:

$$|z|<1.$$

Hence the set $\{Re(w)<0,\quad\forall w\in\mathbb{C}\}$ maps to the set $\{|z|<1,\quad\forall z\in\mathbb{C}\}$ through (2) for $a=c=d=1$, and $b=-1$, i.e.

$$s=\frac{z-1}{z+1}.$$

Remark 1.1. If $Re(s) < 0$ holds for all finite eigenvalues of $sE - A$, then system (1) is asymptotically stable. However, from the above discussion we can conclude that by using the linear fractional transformation for $a = c = d = 1$, $b = -1$, system (1) is asymptotically stable if for all finite eigenvalues of $zF - G$, $|z| < 1$ holds, where $zF - G$ is the pencil of (3), and

$$z = -\frac{s+1}{s-1}. \tag{4}$$

Remark 1.2. To sum up the steps of the mathematical formulation so far, firstly we assumed the singular system (1), then we applied the transform (2) for $a = c = d$, $b = -1$ an arrived at system (3). The Möbius transformation is used for the particular choice of parameters a, b, c, d because this will allow us in Section 3 to study the stability of system (1) and its robustness through the set $\{|z| < 1, \quad \forall z \in \mathbb{C}\}$ instead of the set $\{Re(w) < 0, \quad \forall w \in \mathbb{C}\}$.

Remark 1.3. The contributions of the article are the following. In Section 2 we prove that the right and left eigenvectors of the pencil of system (3) can be explicitly represented by only having insight on the eigenvectors of the pencil of (1), and without resorting to any further processes of computations. In Section 3 we provide our main results, whereby using a linear fractional transformation and matrix pencil techniques we provide a practical test for (1) and its robustness by requiring only the knowledge of the invariants of its pencil. Finally, numerical examples, including practical applications, are given in Section 4 to support our theory.

Throughout the paper, with 0_{ij} we will denote the zero matrix of i rows and j columns, with ^{T} the transpose tensor, and with I_m the identity matrix $m \times m$. Finally, let $B_{n_1} \in \mathbb{C}^{n_1 \times n_1}$, $B_{n_2} \in \mathbb{C}^{n_2 \times n_2}, \ldots, B_{n_r} \in \mathbb{C}^{n_r \times n_r}$. With the direct sum

$$B_{n_1} \oplus B_{n_2} \oplus \cdots \oplus B_{n_r}$$

we will denote the block diagonal matrix:

$$\text{blockdiag}\begin{bmatrix} B_{n_1} & B_{n_2} & \ldots & B_{n_r} \end{bmatrix}.$$

2. Mathematical background and notation

This section introduces some preliminary concepts and definitions from matrix pencil theory, which are used throughout the paper. The connection between (1), and (3), i.e. between their matrix pencils $sE - A$, $zF - G$, is a consequence of the linear fractional transformation. The notions of their relation are qualified algebraically in terms of relationships between the strict equivalence invariants of the associated pencils. These relationships are summarized below. Let s_j, $j = 0, 1, \ldots, \nu$, and z_j, $j = 0, 1, \ldots, \tilde{\nu}$ be finite eigenvalue: For $a, c, d = 1, \ b = -1$:

- If $s \longrightarrow 0$ then $z \longrightarrow -\frac{b}{a} = 1$;
- If $s \longrightarrow \infty$ then $z \longrightarrow -\frac{d}{c} = -1$;
- If $s \longrightarrow s_j$ then $z \longrightarrow \frac{-ds_j+b}{cs_j-a} = \frac{-s_j-1}{s_j-1}$;
- If $z \longrightarrow 0$ then $s \longrightarrow \frac{b}{d} = -1$;
- If $z \longrightarrow \infty$ then $s \longrightarrow \frac{a}{c} = 1$;
- If $z \longrightarrow z_j$ then $s \longrightarrow \frac{az_j+b}{cz_j+d} = \frac{z_j-1}{z_j+1}$;

Hence:

- If $1, \infty$ are eigenvalues of $sE - A$ then $\tilde{\nu} = \nu$;
- If 1 is an eigenvalue of $sE - A$ but ∞ is not then $\tilde{\nu} = \nu - 1$;
- If 1 is not an eigenvalue of $sE - A$ but ∞ is then $\tilde{\nu} = \nu + 1$.

As already mentioned in the previous section, in this article we assume that the pencil is *regular*. Then invariants of the following type are possible to exist:

- An eigenvalue equal to 1 of algebraic multiplicity p_0;
- ν finite eigenvalues $s_j \neq 1$ of algebraic multiplicity p_j, $j = 1, 2, \ldots, \nu$, $\sum_{j=0}^{\nu} p_j = p$;
- an infinite eigenvalue of algebraic multiplicity q; where $p + q = m$.

If the pencil $sE - A$ of system (1) is regular, then from its regularity there exist non-singular matrices $P, Q \in \mathbb{C}^{m \times m}$ such that

$$PEQ = I_{p_0} \oplus I_p \oplus H_q, \quad PAQ = J_{p_0} \oplus J_p \oplus I_q, \tag{5}$$

where $p_0 + p + q = m$, $J_{p_0} \in \mathbb{C}^{p_0 \times p_0}$, $J_p \in \mathbb{C}^{p \times p}$ are Jordan matrices related to the eigenvalue that is equal to 1 and of the rest of the finite eigenvalues respectively, see [10,14], and $H_q \in \mathbb{C}^{q \times q}$ is a nilpotent matrix with index q_*, constructed by using the algebraic multiplicity of the infinite eigenvalue. The matrices P and Q contain the eigenvectors of $sE - A$. Let

$$P = \begin{bmatrix} P_{p_0} \\ P_p \\ P_q \end{bmatrix}, \quad Q = \begin{bmatrix} Q_{p_0} & Q_p & Q_q \end{bmatrix},$$

with $P_{p_0} \in \mathbb{C}^{p_0 \times m}$, $P_p \in \mathbb{C}^{p \times m}$, $P_q \in \mathbb{C}^{q \times m}$, and, $Q_{p_0} \in \mathbb{C}^{m \times p_0}$, $Q_p \in \mathbb{C}^{m \times p}$, $Q_q \in \mathbb{C}^{m \times q}$, where P_{p_0}, P_p, P_q are matrices with rows the left eigenvectors of eigenvalue 1, rest of the finite eigenvalues, the infinite eigenvalue respectively, where Q_{p_0}, Q_p, Q_q are matrices with columns the right eigenvectors of eigenvalue 1, rest of the finite eigenvalues, the infinite eigenvalue respectively.

We consider now system (3). Then invariants of the following type are possible to exist:

- $\tilde{\nu} + 1$ finite eigenvalues of algebraic multiplicity $\tilde{p}_j$, $j = 0, 1, \dots, \tilde{\nu}$;
- an infinite eigenvalue of algebraic multiplicity $\tilde{q}$.

where $\sum_{j=0}^{\tilde{\nu}} \tilde{p}_j = \tilde{p}$ and $\tilde{p} + \tilde{q} = m$. For a *regular pencil*, there exist non-singular matrices $\tilde{P}$, $\tilde{Q} \in \mathbb{C}^{m \times m}$ such that:

$$\tilde{P}F\tilde{Q} := F_w = I_{\tilde{p}} \oplus \tilde{H}_{\tilde{q}}, \quad \tilde{P}G\tilde{Q} := G_w = \tilde{J}_{\tilde{p}} \oplus I_{\tilde{q}}. \tag{6}$$

Let

$$\tilde{P} = \begin{bmatrix} \tilde{P}_{\tilde{p}} \\ \tilde{P}_{\tilde{q}} \end{bmatrix} \quad \tilde{Q} = \begin{bmatrix} \tilde{Q}_{\tilde{p}} & \tilde{Q}_{\tilde{q}} \end{bmatrix},$$

with $\tilde{P}_{\tilde{p}} \in \mathbb{C}^{\tilde{p} \times m}$, $\tilde{P}_{\tilde{q}} \in \mathbb{C}^{\tilde{q} \times m}$, and $\tilde{Q}_{\tilde{p}} \in \mathbb{C}^{m \times \tilde{p}}$, $\tilde{Q}_{\tilde{q}} \in \mathbb{C}^{m \times \tilde{q}}$, where $\tilde{P}_{\tilde{p}}, \tilde{P}_{\tilde{q}}$ are matrices with rows the left eigenvectors of the finite eigenvalues, and the infinite eigenvalue respectively, where $\tilde{Q}_{\tilde{p}}, \tilde{Q}_{\tilde{q}}$ are matrices with columns the right eigenvectors of the finite eigenvalues, and the infinite eigenvalue respectively.

Theorem 2.1. *We consider the systems (1), and (3). Let P, be the matrix whose rows are the m linear independent eigenvectors of $sE - A$, and Q be the matrix whose columns are the m linear independent eigenvectors of $sE - A$. Furthermore, let $\tilde{P}$, be the matrix whose rows are the m linear independent eigenvectors of $sF - G$, and $\tilde{Q}$ be the matrix whose columns are the m linear independent eigenvectors of $sF - G$. The matrices P, Q are defined in (5), and the matrices $\tilde{P}$, $\tilde{Q}$ are defined in (6). Then:*

$$\tilde{P}_{\tilde{p}} = \begin{bmatrix} P_p \\ P_q \end{bmatrix}, \quad \tilde{Q}_{\tilde{p}} = \begin{bmatrix} Q_p & Q_q \end{bmatrix}. \tag{7}$$

Proof. We consider P, Q as defined in (5). By substituting the transformation

$$Y(t) = QZ(t).$$

into (1), and by multiplying by P we obtain

$$PEQZ'(t) = PAQZ(t).$$

Then by taking the form of Q in (5), and setting

$$Z(t) = \begin{bmatrix} Z_{p_0}(t) \\ Z_p(t) \\ Z_q(t) \end{bmatrix},$$

with $Z_{p_0}(t) \in \mathbb{C}^{p_0 \times 1}$, $Z_p(t) \in \mathbb{C}^{p \times 1}$, $Z_q(t) \in \mathbb{C}^{q \times 1}$, we arrive at three subsystems of (1):

$$Z'_{p_0}(t) = J_{p_0} Z_{p_0}(t);$$

$$Z'_p(t) = J_p Z_p(t);$$

$$H_q Z'_q(t) = Z_q(t).$$

The first two subsystems have solutions:

$$Z_{p_0}(t) = e^{J_{p_0} t} Z_{p_0}(0), \quad Z_p(t) = e^{J_p t} Z_p(0).$$

For the third subsystem let q_* be the index of the nilpotent matrix H_q, i.e. $H_q^{q_*} = 0_{q,q}$. Then if we repeatedly multiply by H_q we obtain the following matrix equations:

$$\begin{aligned} H_q Z_q'(t) &= Z_q(t) \\ H_q^2 Z_q''(t) &= H_q Z_q'(t) \\ H_q^3 Z_q'''(t) &= H_q^2 Z_q''(t) \\ H_q^4 Z_q^{(4)}(t) &= H_q^3 Z_q'''(t) \\ &\vdots \\ H_q^{q_*-1} Z_q^{(q_*-1)}(t) &= H_q^{q_*-2} Z_q^{(q_*-2)}(t) \\ H_q^{q_*} Z_q^{(q_*)}(t) &= H_q^{q_*-1} Z_q^{(q_*-1)}(t) \end{aligned} .$$

By taking the sum of the above equations we arrive at

$$\Big(\sum_{i=1}^{q_*-1} H_q^i Z_q^{(i)}(t)\Big) + H_q^{q_*} Z_q^{(q_*)}(t) = \Big(\sum_{i=1}^{q_*-1} H_q^i Z_q^{(i)}(t)\Big) + Z_q(t),$$

or, equivalently, by taking into account that $H_q^{q_*} = 0_{q,q}$, at the solution:

$$Z_q(t) = 0_{q,1}.$$

By using the solutions of the three subsystems, we obtain:

$$Y(t) = QZ(t) = \begin{bmatrix} Q_{p_0} & Q_p & Q_q \end{bmatrix} \begin{bmatrix} e^{J_{p_0} t} Z_{p_0}(0) \\ e^{J_p t} Z_p(0) \\ 0_{q,1} \end{bmatrix},$$

or, equivalently,

$$Y(t) = Q_{p_0} e^{J_{p_0} t} Z_{p_0}(0) + Q_p e^{J_p t} Z_p(0),$$

or, equivalently,

$$Y(t) = \begin{bmatrix} Q_{p_0} & Q_p \end{bmatrix} e^{J_{p_0+p} t} Z_{p_0+p}(0),$$

where $e^{J_{p_0+p} t} = e^{J_{p_0} t} \oplus e^{J_p t}$, and $Z_{p_0+p}(0) = \begin{bmatrix} Z_{p_0}(0) \\ Z_p(0) \end{bmatrix}$ is a constant vector. This means that $\begin{bmatrix} Q_{p_0} & Q_p \end{bmatrix}$ is the matrix that contains the $p_0 + p$ linear independent eigenvectors of the finite eigenvalues of $sE - A$ which is the pencil of (1). Let us now consider the system (3):

$$F\tilde{Y}' = G\tilde{Y},$$

or, equivalently,

$$(E - A)\tilde{Y}' = (E + A)\tilde{Y}.$$

We apply the transformation

$$\tilde{Y}(t) = Q\tilde{Z}(t).$$

into (3), and multiply by P:

$$P(E - A)Q\tilde{Z}'(t) = P(A + E)Q\tilde{Z}(t),$$

or, equivalently, by using (5):

$$\begin{aligned} &[(I_{p_0} - J_{p_0}) \oplus (I_p - J_p) \oplus (H_q - I_q)]\tilde{Z}'(t) = \\ &[(J_{p_0} + I_{p_0}) \oplus (J_p + I_p) \oplus (I_q + H_q)]\tilde{Z}(t), \end{aligned}$$

whereby setting

$$\tilde{Z}(t) = \begin{bmatrix} \tilde{Z}_{p_0}(t) \\ \tilde{Z}_p(t) \\ \tilde{Z}_q(t) \end{bmatrix},$$

with $\tilde{Z}_{p_0}(t) \in \mathbb{C}^{p_0\times 1}$, $\tilde{Z}_p(t) \in \mathbb{C}^{p\times 1}$, $\tilde{Z}_q(t) \in \mathbb{C}^{q\times 1}$, and using the above written notations we arrive at three subsystems of (3):

$$(I_{p_0} - J_{p_0})\tilde{Z}'_{p_0}(t) = (J_{p_0} + I_{p_0})\tilde{Z}_{p_0}(t);$$

$$(I_p - J_p)\tilde{Z}'_p(t) = (J_p + I_p)\tilde{Z}_p(t);$$

$$(H_q - I_q)\tilde{Z}'_q(t) = (I_q + H_q)\tilde{Z}_q(t).$$

Note that the matrix $I_{p_0} - J_{p_0}$ has only zeros in its diagonal. Furthermore, the matrices $I_p - J_p$, $H_q - I_q$ are both invertible since they are either upper triangular matrices or diagonal with all elements in their diagonal non-zero. The solution of the first subsystem is:

$$\tilde{Z}_{p_0}(t) = 0_{p_0,1}.$$

This can be proved similarly to the relevant part of the proof above for the solution of the system $H_q Z'_q(t) = Z_q(t)$. The two other subsystems have solutions:

$$\tilde{Z}_p(t) = e^{\tilde{J}_p t}\tilde{Z}_p(0), \quad \text{and} \quad \tilde{Z}_q(t) = e^{\tilde{J}_q t}\tilde{Z}_q(0),$$

respectively, where

$$\tilde{J}_p = (I_p - J_p)^{-1}(J_p + I_p), \quad \tilde{J}_q = (H_q - I_q)^{-1}(I_q + H_q).$$

By using the solutions of the three subsystems, and the notation for Q in (5) we obtain:

$$\tilde{Y}(t) = Q\tilde{Z}(t) = \begin{bmatrix} Q_{p_0} & Q_p & Q_q \end{bmatrix} \begin{bmatrix} 0_{p_0,1} \\ e^{\tilde{J}_p t}\tilde{Z}_p(0) \\ e^{\tilde{J}_q t}\tilde{Z}_q(0) \end{bmatrix},$$

or, equivalently,

$$\tilde{Y}(t) = Q_p e^{\tilde{J}_p t}\tilde{Z}_p(0) + Q_q e^{\tilde{J}_q t}\tilde{Z}_q(0),$$

or, equivalently,

$$\tilde{Y}(t) = \begin{bmatrix} Q_p & Q_q \end{bmatrix} e^{J_{p+q} t}\tilde{Z}_{p+q}(0),$$

where $e^{J_{p+q}t} = e^{\tilde{J}_p t} \oplus e^{\tilde{J}_q t}$, $\tilde{Z}_{p+q}(0) = \begin{bmatrix} \tilde{Z}_p(0) \\ \tilde{Z}_q(0) \end{bmatrix}$. This means that $\begin{bmatrix} Q_p & Q_q \end{bmatrix}$ is the matrix that contains all linear independent eigenvectors of the finite eigenvalues of $sF - G$ which is the pencil of (3). Hence

$$\tilde{Q}_{\tilde{p}} = \begin{bmatrix} Q_p & Q_q \end{bmatrix}.$$

Let us now consider the system

$$F^T \hat{Y}'^T = G^T \hat{Y}^T,$$

or, equivalently,

$$\hat{Y}' F = \hat{Y} G,$$

or, equivalently,

$$\hat{Y}'(E - A) = \hat{Y}(E + A),$$

where $\hat{Y} \in \mathbb{C}^{1\times m}$. We apply the transformation

$$\hat{Y}(t) = \hat{Z}(t)P$$

into the above system, and multiply by Q:

$$\hat{Z}'(t)P(E - A)Q = \hat{Z}(t)P(A + E)Q,$$

or, equivalently,

$$\hat{Z}'(t)[(I_{p_0} - J_{p_0}) \oplus (I_p - J_p) \oplus (H_q - I_q)] =$$
$$\hat{Z}(t)[(J_{p_0} + I_{p_0}) \oplus (J_p + I_p) \oplus (I_q + H_q)],$$

whereby setting

$$\hat{Z}(t) = \begin{bmatrix} \hat{Z}_{p_0}(t) & \hat{Z}_p(t) & \hat{Z}_q(t) \end{bmatrix},$$

with $\hat{Z}_{p_0}(t) \in \mathbb{C}^{1\times p_0}$, $\hat{Z}_p(t) \in \mathbb{C}^{1\times p}$, $\hat{Z}_q(t) \in \mathbb{C}^{1\times q}$, and using the above written notations we arrive at three subsystems:

$$\hat{Z}'_{p_0}(t)(I_{p_0} - J_{p_0}) = \hat{Z}_{p_0}(t)(J_{p_0} + I_{p_0});$$

$$\hat{Z}'_p(t)(I_p - J_p) = \hat{Z}_p(t)(J_p + I_p);$$

$$\hat{Z}'_q(t)(H_q - I_q) = \hat{Z}_q(t)(I_q + H_q).$$

As already written the matrix $I_{p_0} - J_{p_0}$ has only zeros in its diagonal. Furthermore, the matrices $I_p - J_p$, $H_q - I_q$ are both invertible since all elements in their diagonal are non-zero. The solution of the first subsystem is

$$\hat{Z}_{p_0}(t) = 0_{1,p_0}.$$

This can be proved similarly to the relevant part of the proof above for the solution of the system $H_q Z'_q(t) = Z_q(t)$. The two other subsystems have solutions:

$$\hat{Z}_p(t) = \hat{Z}_p(0)e^{\hat{J}_p t}, \quad \text{and} \quad \hat{Z}_q(t) = \hat{Z}_q(0)e^{\hat{J}_q t},$$

respectively, where

$$\hat{J}_p = (J_p + I_p)(I_p - J_p)^{-1}, \quad \hat{J}_q = (I_q + H_q)(H_q - I_q)^{-1}.$$

By using the solutions of the three subsystems, and the notation for P as written in (5) we obtain:

$$\hat{Y}(t) = \hat{Z}(t)P = \begin{bmatrix} 0_{p_0,1} & e^{\hat{J}_p t}\hat{Z}_p(0) & e^{\hat{J}_q t}\hat{Z}_q(0) \end{bmatrix} \begin{bmatrix} P_{p_0} \\ P_p \\ P_q \end{bmatrix},$$

or, equivalently,

$$\hat{Y}(t) = \hat{Z}_p(0)e^{\hat{J}_p t}P_p + \hat{Z}_q(0)e^{\hat{J}_q t}P_q,$$

or, equivalently,

$$\hat{Y}(t) = \hat{Z}_{p+q}(0)e^{J_{p+q}t} \begin{bmatrix} P_p \\ P_q \end{bmatrix},$$

where $e^{J_{p+q}t} = e^{\hat{J}_p t} \oplus e^{\hat{J}_q t}$, $\hat{Z}_{p+q}(0) = \begin{bmatrix} \hat{Z}_p(0) & \hat{Z}_q(0) \end{bmatrix}$. This means that $\begin{bmatrix} P_p \\ P_q \end{bmatrix}$ is the matrix that contains the all left linear independent eigenvectors of the finite eigenvalues of $sF - G$ which is the pencil of (3). Hence:

$$\tilde{P}_{\tilde{p}} = \begin{bmatrix} P_p \\ P_q \end{bmatrix}.$$

The proof is complete.

Remark 2.1. In (7) the matrices $\tilde{P}_{\tilde{p}}$, $\tilde{Q}_{\tilde{p}}$ are defined from eigenvectors related to eigenvalues of the pencil $sE - A$ of system (1). Hence, it is worth mentioning that these matrices are not uniquely defined since only the span of the eigenvectors, i.e. eigenspace, is unique; any basis of this is a basis of eigenvectors to the given eigenvalue and may form the rows of the matrix.

3. Robust stability

In this section we present our main result. In general, if we consider the perturbed system of (1):

$$(E + \Delta E)Y'(t) = (A + \Delta A)Y(t), \tag{8}$$

and its pencil $s(E + \Delta E) - (A + \Delta A)$, where $\Delta E, \Delta A \in \mathbb{R}^{r\times m}$, then its eigenvalues are given by solving the following characteristic equation:

$$\left| s(E + \Delta E) - (A + \Delta A) \right| = 0.$$

By applying the transform (2) we get

$$\left| \frac{az + b}{cz + d}(E + \Delta E) - (A + \Delta A) \right| = 0,$$

or, equivalently, by using determinant properties

$$\left|(az+b)(E+\Delta E)-(cz+d)(A+\Delta A)\right|=0,$$

or, equivalently,

$$\left|(a(E+\Delta E)-c(A+\Delta A))z-(d(A+\Delta A)-b(E+\Delta E))\right|=0,$$

or, equivalently,

$$\left|((aE-cA)+(a\Delta E-c\Delta A))z-((dA-bE)+(d\Delta A-b\Delta E))\right|=0,$$

which is the characteristic equation of the system

$$(F+\Delta F)\tilde{Y}'(t)=(G+\Delta G)\tilde{Y}(t),$$

where

$$\Delta F=a\Delta E-c\Delta A,\quad \Delta G=d\Delta A-b\Delta E.$$

We focus on the following perturbed system:

$$EY'(t)=(A+\Delta A)Y(t). \tag{9}$$

Theorem 3.1. *Consider system (1) with a regular pencil, and finite eigenvalues of type s_j, $j=0,1,\dots,\nu$ of algebraic multiplicity p_j. Let this system be asymptotically stable. Then, after a perturbation accordingly to (9), the system will retain its asymptotic stability if for all finite eigenvalues $s_j\in\mathbb{C}$ of the pencil of (1):*

$$\left|\frac{s_j+1}{s_j-1}\right|+2p_*K<1,\quad j=0,1,\dots,\nu.$$

where $K=\frac{\|P\Delta A\|_1}{\|Q^{-1}\|_1}$, $p_*=max_{1\leq j\leq\nu}p_j$ and P, Q are defined in (5).

Proof. We consider (1), and the perturbed system (9). If we apply the linear fractional transformation (2) to these two systems for $a=c=d=1$, $b=-1$, we arrive at system (3), and its perturbed system (8) for $\Delta E=0_{m,m}$ with z_j, $j=0,1,\dots,\tilde{\nu}$, being a finite eigenvalue of the pencil $zF-G$, and $\tilde{z}$ a random finite eigenvalue of the pencil of the perturbed system. Let $U\in\mathbb{C}^{m\times 1}$ be an eigenvector of $\tilde{z}$, i.e. $\tilde{z}(F-\Delta A)U=(G+\Delta A)U$, or, equivalently,

$$(\tilde{z}F-G)U=\tilde{E}U.$$

where $\tilde{E}=(\tilde{z}+1)\Delta A$. We consider the matrices $\tilde{P}$, $\tilde{Q}$ as defined in (6). By substituting the transformation $U=\tilde{Q}W$ into the above expression, $W\in\mathbb{C}^{m\times 1}$, we obtain

$$(\tilde{z}F-G)\tilde{Q}W=\tilde{E}\tilde{Q}W,$$

whereby, multiplying by $\tilde{P}$, and taking into account the definition of F_w, G_w in (6), we get

$$(\tilde{z}F_w-G_w)W=\tilde{P}\tilde{E}\tilde{Q}W.$$

While $\tilde{z}\neq z_j$ we have $\det(\tilde{z}F_w-G_w)\neq 0$ and

$$W=(\tilde{z}F_w-G_w)^{-1}\tilde{P}\tilde{E}\tilde{Q}W.$$

Thus

$$(\tilde{z}F_w-G_w)^{-1}=\begin{bmatrix}(\tilde{z}I_{\tilde{p}}-\tilde{J}_{\tilde{p}})^{-1} & 0_{\tilde{p},\tilde{q}}\\ 0_{\tilde{q},\tilde{p}} & (\tilde{z}H_{\tilde{q}}-I_{\tilde{q}})^{-1}\end{bmatrix}.$$

By using (6), the matrix $\tilde{P}\tilde{E}\tilde{Q}$ can be written as

$$\tilde{P}\tilde{E}\tilde{Q}=\begin{bmatrix}\tilde{P}_{\tilde{p}}\\ \tilde{P}_{\tilde{q}}\end{bmatrix}\tilde{E}\begin{bmatrix}\tilde{Q}_{\tilde{p}} & \tilde{Q}_{\tilde{q}}\end{bmatrix}=\begin{bmatrix}\tilde{P}_{\tilde{p}}\tilde{E}\tilde{Q}_{\tilde{p}} & \tilde{P}_{\tilde{p}}\tilde{E}\tilde{Q}_{\tilde{q}}\\ \tilde{P}_{\tilde{q}}\tilde{E}\tilde{Q}_{\tilde{p}} & \tilde{P}_{\tilde{q}}\tilde{E}\tilde{Q}_{\tilde{q}}\end{bmatrix}.$$

Moreover, let $W=\begin{bmatrix}W_{\tilde{p}}\\ W_{\tilde{q}}\end{bmatrix}$, $W_{\tilde{p}}\in\mathbb{C}^{\tilde{p}\times m}$, $W_{\tilde{q}}\in\mathbb{C}^{\tilde{q}\times m}$. Then

$$\begin{bmatrix}W_{\tilde{p}}\\ W_{\tilde{q}}\end{bmatrix}=\begin{bmatrix}(\tilde{z}I_{\tilde{p}}-\tilde{J}_{\tilde{p}})^{-1} & 0_{\tilde{p},\tilde{q}}\\ 0_{\tilde{q},\tilde{p}} & (\tilde{z}H_{\tilde{q}}-I_{\tilde{q}})^{-1}\end{bmatrix}\begin{bmatrix}\tilde{P}_{\tilde{p}}\tilde{E}\tilde{Q}_{\tilde{p}} & \tilde{P}_{\tilde{p}}\tilde{E}\tilde{Q}_{\tilde{q}}\\ \tilde{P}_{\tilde{q}}\tilde{E}\tilde{Q}_{\tilde{p}} & \tilde{P}_{\tilde{q}}\tilde{E}\tilde{Q}_{\tilde{q}}\end{bmatrix}\begin{bmatrix}W_{\tilde{p}}\\ W_{\tilde{q}}\end{bmatrix}.$$

From the above equation we get

$$W_{\tilde p} = (\tilde z I_{\tilde p} - \tilde J_{\tilde p})^{-1} \begin{bmatrix} \tilde P_{\tilde p}\tilde E\tilde Q_{\tilde p} & \tilde P_{\tilde p}\tilde E\tilde Q_{\tilde q} \end{bmatrix} \begin{bmatrix} W_{\tilde p} \\ W_{\tilde q} \end{bmatrix},$$

which can be written as

$$\begin{bmatrix} W_{\tilde p} \\ 0_{\tilde q,1} \end{bmatrix} = \begin{bmatrix} (\tilde z I_{\tilde p} - \tilde J_{\tilde p})^{-1} & 0_{\tilde p,\tilde q} \\ 0_{\tilde q,\tilde p} & 0_{\tilde q,\tilde q} \end{bmatrix} \begin{bmatrix} \tilde P_{\tilde p}\tilde E\tilde Q_{\tilde p} & \tilde P_{\tilde p}\tilde E\tilde Q_{\tilde q} \\ 0_{\tilde q,\tilde p} & 0_{\tilde q,\tilde q} \end{bmatrix} \begin{bmatrix} W_{\tilde p} \\ W_{\tilde q} \end{bmatrix}.$$

Let Q_1, Q_2 be left inverses of $\tilde Q_{\tilde p}$, $\tilde Q_{\tilde q}$ respectively. From the transformation $U = \tilde Q W$ we get $\begin{bmatrix} Q_1 U \\ Q_2 U \end{bmatrix} = \begin{bmatrix} W_{\tilde p} \\ W_{\tilde q} \end{bmatrix}$ and thus $Q_1 U = W_{\tilde p}$. Hence by using this observation into the above expression we get

$$\begin{bmatrix} Q_1 \\ 0_{\tilde q,m} \end{bmatrix} U = \begin{bmatrix} (\tilde z I_{\tilde p} - \tilde J_{\tilde p})^{-1} & 0_{\tilde p,\tilde q} \\ 0_{\tilde q,\tilde p} & 0_{\tilde q,\tilde q} \end{bmatrix} \begin{bmatrix} \tilde P_{\tilde p}\tilde E\tilde Q_{\tilde p} & \tilde P_{\tilde p}\tilde E\tilde Q_{\tilde q} \\ 0_{\tilde q,\tilde p} & 0_{\tilde q,\tilde q} \end{bmatrix} \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} U.$$

Furthermore

$$\left\| \begin{bmatrix} Q_1 \\ 0_{m,\tilde q} \end{bmatrix} U \right\| = \left\| \begin{bmatrix} (\tilde z I_{\tilde p} - \tilde J_{\tilde p})^{-1} & 0_{\tilde p,\tilde q} \\ 0_{\tilde q,\tilde p} & 0_{\tilde q,\tilde q} \end{bmatrix} \begin{bmatrix} \tilde P_{\tilde p}\tilde E\tilde Q_{\tilde p} & \tilde P_{\tilde p}\tilde E\tilde Q_{\tilde q} \\ 0_{\tilde q,\tilde p} & 0_{\tilde q,\tilde q} \end{bmatrix} \tilde Q^{-1} U \right\|$$

and

$$sup \left\{ \frac{\left\| \begin{bmatrix} Q_1 \\ 0_{m,\tilde q} \end{bmatrix} U \right\|}{\|U\|} \right\} = sup \left\{ \frac{\left\| \begin{bmatrix} (\tilde z I_{\tilde p} - \tilde J_{\tilde p})^{-1} & 0_{\tilde p,\tilde q} \\ 0_{\tilde q,\tilde p} & 0_{\tilde q,\tilde q} \end{bmatrix} \begin{bmatrix} \tilde P_{\tilde p}\tilde E\tilde Q_{\tilde p} & \tilde P_{\tilde p}\tilde E\tilde Q_{\tilde q} \\ 0_{\tilde q,\tilde p} & 0_{\tilde q,\tilde q} \end{bmatrix} \tilde Q^{-1} U \right\|}{\|U\|} \right\},$$

since $U \neq 0_{m,1}$ is an eigenvector of $\tilde z$. Hence

$$\left\| \begin{bmatrix} Q_1 \\ 0_{\tilde q,m} \end{bmatrix} \right\| = \left\| \begin{bmatrix} (\tilde z I_{\tilde p} - \tilde J_{\tilde p}(z_j))^{-1} & 0_{\tilde p,\tilde q} \\ 0_{\tilde q,\tilde p} & 0_{\tilde q,\tilde q} \end{bmatrix} \begin{bmatrix} \tilde P_{\tilde p}\tilde E\tilde Q_{\tilde p} & \tilde P_{\tilde p}\tilde E\tilde Q_{\tilde q} \\ 0_{\tilde q,\tilde p} & 0_{\tilde q,\tilde q} \end{bmatrix} \tilde Q^{-1} \right\|,$$

where

$$\begin{bmatrix} \tilde P_{\tilde p}\tilde E\tilde Q_{\tilde p} & \tilde P_{\tilde p}\tilde E\tilde Q_{\tilde q} \\ 0_{\tilde q,\tilde p} & 0_{\tilde q,\tilde q} \end{bmatrix} = \begin{bmatrix} \tilde P_{\tilde p}\tilde E \\ 0_{\tilde q,m} \end{bmatrix} \begin{bmatrix} \tilde Q_{\tilde p} & \tilde Q_{\tilde q} \end{bmatrix} = \begin{bmatrix} \tilde P_{\tilde p}\tilde E \\ 0_{\tilde q,m} \end{bmatrix} \tilde Q.$$

Thus

$$\left\| \begin{bmatrix} Q_1 \\ 0_{\tilde q,m} \end{bmatrix} \right\| = \left\| \begin{bmatrix} (\tilde z I_{\tilde p} - \tilde J_{\tilde p}(z_j))^{-1} & 0_{\tilde p,\tilde q} \\ 0_{\tilde q,\tilde p} & 0_{\tilde q,\tilde q} \end{bmatrix} \begin{bmatrix} \tilde P_{\tilde p}\tilde E \\ 0_{\tilde q,m} \end{bmatrix} \right\|,$$

or, equivalently,

$$\left\| \begin{bmatrix} Q_1 \\ 0_{\tilde q,m} \end{bmatrix} \right\| = \left\| \begin{bmatrix} (\tilde z I_{\tilde p} - \tilde J_{\tilde p})^{-1}\tilde P_{\tilde p}\tilde E \\ 0_{\tilde q,m} \end{bmatrix} \right\|.$$

or, equivalently,

$$\left\| \begin{bmatrix} Q_1 \\ 0_{\tilde q,m} \end{bmatrix} \right\| \leq \left\| (\tilde z I_{\tilde p} - \tilde J_{\tilde p})^{-1} \right\| \left\| \begin{bmatrix} \tilde P_{\tilde p}\tilde E \\ 0_{\tilde q,m} \end{bmatrix} \right\|,$$

or, equivalently,

$$\frac{1}{\left\| (\tilde z I_{\tilde p} - \tilde J_{\tilde p})^{-1} \right\|} \leq \frac{\left\| \begin{bmatrix} \tilde P_{\tilde p}\tilde E \\ 0_{\tilde q,m} \end{bmatrix} \right\|}{\left\| \begin{bmatrix} Q_1 \\ 0_{\tilde q,m} \end{bmatrix} \right\|},$$

where $(\tilde z I_{\tilde p} - \tilde J_{\tilde p})^{-1} := [\tilde z I_{\tilde p_1} - \tilde J_{\tilde p_1}(\tilde z_1)]^{-1} \oplus [\tilde z I_{\tilde p_2} - \tilde J_{\tilde p_2}(\tilde z_2)]^{-1} \oplus \cdots \oplus [\tilde z I_{\tilde p_{\tilde \nu}} - \tilde J_{\tilde p_{\tilde \nu}}(\tilde z_{\tilde \nu})]^{-1}$ and, see [15]:

$$[\tilde z I_{\tilde p_j} - \tilde J_{\tilde p_j}(z_j)]^{-1} = \begin{bmatrix} \frac{1}{\tilde z - z_j} & \frac{1}{(\tilde z - z_j)^2} & \cdots & \frac{1}{(\tilde z - z_j)^{\tilde p_j}} \\ 0 & \frac{1}{\tilde z - z_j} & \cdots & \frac{1}{(\tilde z - z_j)^{\tilde p_j - 1}} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{1}{\tilde z - z_j} \end{bmatrix}, \quad \forall j = 0, 1, \ldots, \tilde \nu.$$

We have assumed asymptotic stability for (1), i.e. the real part of each finite eigenvalue of the pencil of (1) is negative, and consequently $|z_j| < 1$. Let $M_j = \frac{1}{|\tilde{z}-z_j|}$. By taking the norm $\|\cdot\|_1$ of $(\tilde{z}I_{\tilde{p}} - \tilde{J}_{\tilde{p}})^{-1}$ we have

$$\left\|[\tilde{z}I_{\tilde{p}} - \tilde{J}_{\tilde{p}}]^{-1}\right\|_1 = \max_{1\le j\le\tilde{\nu}} \sum_{l=1}^{\tilde{p}_j} \frac{1}{\left|\tilde{z}-z_j\right|^l} \le \max_j M_j \tilde{p}_j \le (\max_{1\le j\le\tilde{\nu}} M_j)p_*.$$

where $max_{1\le j\le\nu} p_j = p_*$. Equivalently:

$$\frac{1}{(max_{1\le j\le\tilde{\nu}} M_j)p_*} \le \frac{1}{\left\|[\tilde{z}I_{\tilde{p}} - \tilde{J}_{\tilde{p}}]^{-1}\right\|_1},$$

or, equivalently,

$$\frac{min_{1\le j\le\tilde{\nu}}\left|\tilde{z}-z_j\right|}{p_*} \le \frac{1}{\left\|[\tilde{z}I_{\tilde{p}} - \tilde{J}_{\tilde{p}}]^{-1}\right\|_1},$$

or, equivalently,

$$\frac{min_{1\le j\le\tilde{\nu}}\left|\tilde{z}-z_j\right|}{p_*} \le \frac{\left\|\begin{bmatrix} \tilde{P}_{\tilde{p}}\tilde{E} \\ 0_{\tilde{q},m}\end{bmatrix}\right\|_1}{\left\|\begin{bmatrix} Q_1 \\ 0_{\tilde{q},m}\end{bmatrix}\right\|_1}.$$

From (7) we have that $\tilde{P}_{\tilde{p}} = \begin{bmatrix} P_p \\ P_q \end{bmatrix}$, and $\tilde{Q}_{\tilde{p}} = \begin{bmatrix} Q_p & Q_q \end{bmatrix}$. However, as already written, the real part of each finite eigenvalue of the pencil of (1) is negative and consequently $\tilde{P}_{\tilde{p}} = P$, $\tilde{Q}_{\tilde{p}} = Q$ since the system does not have an eigenvalue equal to 1. Hence

$$\min_{1\le j\le\tilde{\nu}} \left|\tilde{z}-z_j\right| \le \tilde{p}_* \frac{\left\|P\tilde{E}\right\|_1}{\left\|Q^{-1}\right\|_1}.$$

Note that the norm $\|\cdot\|_1$ can be replaced with $\|\cdot\|_\infty$.

Let $\tilde{z}_*$, z_* be eigenvalues such that $\left|\tilde{z}_*\right| = max\left|\tilde{z}\right|$, $z_* = max_{1\le j\le\tilde{\nu}}\left|z_j\right|$. Then

$$\left|\left|\tilde{z}_*\right| - z_*\right| \le p_*(1+\left|\tilde{z}_*\right|)K,$$

or, equivalently,

$$\left|z_j\right| - p_*(1+\left|\tilde{z}_*\right|)K \le \left|\tilde{z}_*\right| \le \left|z_j\right| + p_*(1+\left|\tilde{z}_*\right|)K.$$

The right inequality can be written as

$$\left|\tilde{z}_*\right|(1-p_*K) \le \left|z_j\right| + p_*K,$$

or, equivalently,

$$\left|\tilde{z}_*\right| \le \frac{\left|z_j\right| + p_*K}{1-p_*K}.$$

Thus if $\frac{\left|z_j\right|+p_*K}{1-p_*K} < 1$, or, equivalently,

$$\left|\frac{s_j+1}{s_j-1}\right| + 2p_*K < 1, \quad j = 0, 1, \ldots, \nu,$$

then $\left|\tilde{z}_*\right| < 1$, i.e. the pencil of the perturbed system (9) will have all its finite eigenvalues with negative real part. The proof is complete.

Remark 3.1. There have been many significant studies on the stability of systems of differential equations, including cases with singularity problems, see [1,4,12,16,17]. The stability test in Theorem 3.1 is practical and easy to use and requires insight on the spectrum of the pencil.

Remark 3.2. To apply the condition proposed in Theorem 3.1, knowledge of the spectrum of the pencil $sE - A$ is required. The standard numerical method for solving the corresponding generalized eigenvalue problem is the QZ algorithm, see [18], which is known to have a computational complexity of $O(n^3)$ floating point operations, and works with dense matrices.

Theorem 3.1 can be further developed by using methods that measure the participation of system eigenvalues in system states, see [19]. In addition it can be used to further develop robust stability tests for singular linear systems of fractional differential equations, see [20,21], other type of fractional operators such as the fractional nabla applied to difference equations, see [22], and other type of non-linear advanced differential equations, see [23,24].

4. Numerical examples

In this section we first apply the main results of this article to a small linear singular system of differential equations. Then, we further exploit our theory with a simple application in electric power engineering. In particular, we consider a synchronous generator connected to a bus of constant frequency and voltage.

4.1. Numerical example 1

We consider system (1) with

$$E = \begin{bmatrix} 6 & -3 & -4 & 0 & 0 \\ 2 & 1 & 1 & 3 & 0 \\ 0 & -4 & -7 & 1 & 0 \\ 4 & 1 & 0 & 7 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \; A = \begin{bmatrix} 5 & 4.5 & 5 & 5.5 & 3 \\ 4 & -2 & -2 & -5 & 2 \\ 9 & 3.5 & 5 & 5.5 & 1 \\ -2 & -3.5 & -3 & -8.5 & 0 \\ 15 & 0 & 0 & 0 & 5 \end{bmatrix}.$$

The matrix pencil $sE - A$ has 3 finite eigenvalues $s_1 = -0.5$, $s_2 = -1$, $s_3 = -2$, and 2 infinite eigenvalues s_4, s_5. Since all finite eigenvalues have algebraic multiplicity equal to 1, we have $p_* = 1$. If u_j, w_j are the right and left eigenvector, respectively, associated with the eigenvalue s_j, then we have:

$$u_1 = \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \; u_2 = \begin{bmatrix} -0.25 \\ 1 \\ -1 \\ 0 \\ 0.75 \end{bmatrix}, \; u_3 = \begin{bmatrix} 0 \\ 1 \\ -0.5 \\ 0 \\ 0 \end{bmatrix}, \; u_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \; u_5 = \begin{bmatrix} -0.1 \\ -1 \\ 0.6 \\ 0.2 \\ -0.19 \end{bmatrix},$$

and

$$w_1 = \begin{bmatrix} -0.38 \\ -1 \\ -0.23 \\ 0 \\ 0.68 \end{bmatrix}^T, \; w_2 = \begin{bmatrix} -0.6 \\ -1 \\ 0.2 \\ 0 \\ 0.72 \end{bmatrix}^T, \; w_3 = \begin{bmatrix} -0.33 \\ 1 \\ 0.11 \\ 0 \\ -0.22 \end{bmatrix}^T, \; w_4 = \begin{bmatrix} 0.03 \\ -1 \\ -0.16 \\ 0.45 \\ -0.34 \end{bmatrix}^T, \; w_5 = \begin{bmatrix} 0.03 \\ -1 \\ -0.16 \\ 0.45 \\ 0.41 \end{bmatrix}^T.$$

Therefore, the matrices Q and P are as follows:

$$Q = \begin{bmatrix} 0 & -0.25 & 0 & 0 & -0.1 \\ -1 & 1 & 1 & 0 & 1-1 \\ 1 & -1 & -0.5 & 0 & 0.6 \\ 0 & 0 & 0 & 0 & 0.2 \\ 0 & 0.75 & 0 & -1 & -0.19 \end{bmatrix}, \; P = \begin{bmatrix} -0.38 & -1 & -0.23 & 0 & 0.68 \\ -0.6 & -1 & 0.2 & 0 & 0.72 \\ -0.33 & 1 & 0.11 & 0 & -0.22 \\ 0.03 & -1 & -0.16 & 0.45 & -0.34 \\ 0.03 & -1 & -0.16 & 0.45 & 0.41 \end{bmatrix}.$$

4.1.1. Case 1

Consider a perturbation of A to be given by the following matrix:

$$\Delta A = \begin{bmatrix} 0 & 0.09 & 0 & 0.11 & 0.06 \\ 0 & 0 & -0.04 & 0 & 0.04 \\ 0 & 0.07 & 0 & 0.11 & 0.02 \\ -0.04 & 0 & -0.06 & 0 & 0 \\ 0.30 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The perturbed right-hand side coefficient matrix is thus:

$$A + \Delta A = \begin{bmatrix} 5 & 4.59 & 5 & 5.61 & 3.06 \\ 4 & -2 & -2.04 & -5 & 2.04 \\ 9 & 3.57 & 5 & 5.61 & 1.02 \\ -2.04 & -3.50 & -3.06 & -8.5 & 0 \\ 15.30 & 0 & 0 & 0 & 5 \end{bmatrix}.$$

From Theorem 3.1, K is determined as follows:

$$K = \frac{\|P\Delta A\|_1}{\|Q^{-1}\|_1} = 0.0432 \,,$$

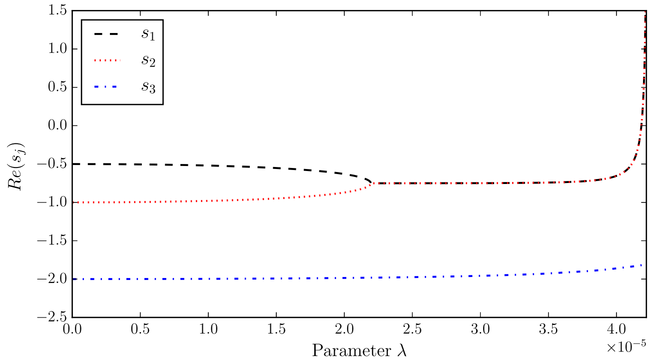

Fig. 1. Real part of finite eigenvalues as parameter λ varies.

where

$$P\Delta A = \begin{bmatrix} 0.22 & -0.04 & 0.04 & -0.04 & -0.07 \\ 0.20 & -0.05 & 0.04 & -0.07 & -0.07 \\ -0.12 & -0.01 & 0.01 & -0.01 & -0.04 \\ 0.11 & -0.01 & 0.01 & -0.01 & -0.04 \\ -0.07 & -0.02 & -0.04 & -0.02 & 0.02 \end{bmatrix},$$

$$Q^{-1} = \begin{bmatrix} -4 & 0 & 0 & -2 & 0 \\ -4 & 1 & 2 & -3 & 0 \\ -3 & 0 & 0 & -2.47 & -1 \\ 0 & 0 & 0 & 5 & 0 \\ 0 & 2 & 2 & 4 & 0 \end{bmatrix}.$$

The stability condition, as derived in Theorem 3.1, is checked here for all the finite eigenvalues of the system. Note that, for the sake of simplicity, in the remaining of this section we will use the notation $C(s_j) = \left|\frac{s_j+1}{s_j-1}\right| + 2p_*K$. The values of $C(s_j)$ for $j = 1, 2, 3$, are as follows:

- For $s_1 = -0.5$ we have
$$C(s_1) = 0.4198\ .$$
- For $s_2 = -1$ we have
$$C(s_2) = 0.0865\ .$$
- For $s_3 = -2$ we have
$$C(s_3) = 0.4198\ .$$

Since $C(s_j) < 1$ holds for all the finite eigenvalues, the perturbed system is stable, according to Theorem 3.1. Indeed, carrying the eigenvalue analysis confirms that the perturbed system is stable. In particular, the pencil $sE - (A + \Delta A)$ has 3 finite eigenvalues $\hat{s}_1 = -0.433$, $\hat{s}_2 = -1.117$, $\hat{s}_3 = -1.999$, and 2 infinite eigenvalues $\hat{s}_4$, $\hat{s}_5$.

4.1.2. Case 2

Consider now the perturbation of A to be given by the following matrix:

$$\Delta A = \begin{bmatrix} 0 & -4.5\lambda & 0 & 0 & 0 \\ 0 & 0 & 2\lambda & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

where λ is a parameter. Obviously, for $\lambda = 0$, we get a zero perturbation matrix. A ramp of λ is simulated until an eigenvalue is past the imaginary axis and the system is driven to instability. The real parts of the system finite eigenvalues with respect to the parameter λ are depicted in Fig. 1. The value of $C(s_j)$ as the value of λ increases is shown in Fig. 2.

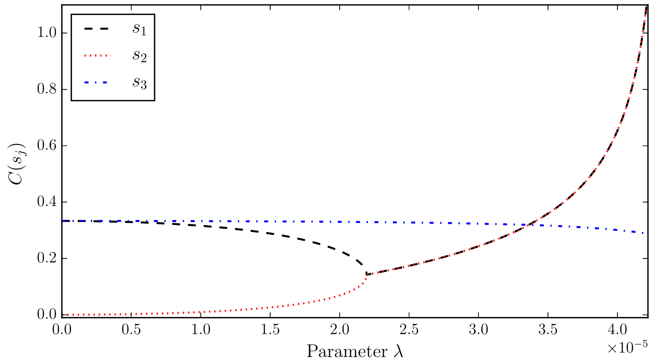

Fig. 2. $C(s_j)$ of finite eigenvalues as parameter λ varies.

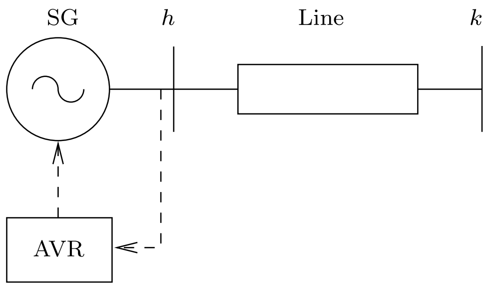

Fig. 3. One machine, two-bus power system.

Since the real part of s_3 is kept negative, we have that $C(s_3) < 1$. For s_1, s_2, we have $C(s_1) > 1$, $C(s_2) > 1$ only when the real parts $Re(s_1)$, $Re(s_2)$, become positive.

4.2. Numerical example 2

In transient stability analysis,[1] a power system model is formulated as a set of non-linear differential–algebraic equations [26]. The stability of a stationary point can be examined by carrying out an eigenvalue analysis of the linearized system around this point. Power systems are, in particular, a relevant application of the theorems presented in this paper, since a linearized power system can be described as a singular system in the form of (1). Note also that linear differential–algebraic equations are just a special case of singular systems in the form of (1) for $E = I_p \oplus 0_{q,q}$. The scheme of the power system considered in this example is depicted in Fig. 3. It consists of a synchronous generator (SG), which is equipped with an automatic voltage regulator (AVR). The AVR is a feedback control system used to stabilize the generator's voltage by adjusting the current in its rotor field winding. Finally, the SG is connected to a bus of constant frequency and voltage, through a transmission line. The non-linear equations of this system are as follows:

- SG (electro)mechanical equations:

$$\dot{\delta}/\Omega_b = \omega - \omega_s \ ,$$

$$2H\dot{\omega} = \tau_m - \tau_e \ ,$$

$$0 = -\tau_e + \psi_d i_q - \psi_q i_d \ ,$$

where δ is the rotor angle; ω the angular speed; τ_e the electrical torque; i_d, i_q are the direct (d-) axis and quadrature (q-) axis current; ψ_q, ψ_d the d-axis and q-axis magnetic flux, respectively. Note that d- is the axis of the machine's rotor dc magnetic field winding, while the axis q- is 90 electrical degrees ahead with respect to the d-axis. In addition,

[1] "Transient stability analysis" refers to the study of the ability of power systems to maintain synchronism and recover a stationary condition after a perturbation [25].

$\Omega_b = 314.16$ rad/s is the nominal synchronous angular frequency; $\omega_s = 1$ pu (rad/s) (in per unit with respect to Ω_b) the reference angular frequency; $\tau_m = 0.6$ pu (MN·m) the mechanical torque; $H = 2.5$ MWs/MVA the inertia constant.

- SG electrical and magnetic equations:

$$
\begin{aligned}
0 &= \psi_q + v_d \ , \\
0 &= -\psi_d + v_q \ , \\
T'_{d0}\dot{e}'_q &= -e'_q - (x_d - x'_d)i_d + v_f \ , \\
T'_{q0}\dot{e}'_d &= -e'_d + (x_q - x'_q)i_q \ , \\
0 &= v_q - e'_q + x'_d i_d \ , \\
0 &= v_d - e'_d - x'_q i_q \ ,
\end{aligned}
$$

where v_d, v_q the d-axis and q-axis voltage; e'_d, e'_q the d-axis and q-axis transient electromotive force; v_f is the field voltage; $T'_{d0} = 8$ s the d-axis and $T'_{q0} = 0.6$ s the q-axis transient time constant; $x_d = 1.7$ pu (Ω) the d-axis, $x_q = 1.7$ pu (Ω) the q-axis synchronous reactance; $x'_d = 0.3$ pu (Ω) the d-axis, $x'_q = 0.5$ pu (Ω) the q-axis transient reactance.

- SG interface with the network and network equations:

$$
\begin{aligned}
0 &= -p_h + v_d i_d + v_q i_q \ , \\
0 &= -q_h + v_q i_d - v_d i_q \ , \\
0 &= v_h \sin(\delta - \theta_h) - v_d \ , \\
0 &= v_h \cos(\delta - \theta_h) - v_q \ , \\
0 &= -p_h + v_h^2 g_L - v_h v_k (g_L \cos\theta_{hk} + b_L \sin\theta_{hk}) \ , \\
0 &= -q_h - v_h^2 b_L - v_h v_k (g_L \sin\theta_{hk} - b_L \cos\theta_{hk}) \ , \\
0 &= -p_k + v_k^2 g_L - v_h v_k (g_L \cos\theta_{hk} - b_L \sin\theta_{hk}) \ , \\
0 &= -q_k - v_k^2 b_L - v_h v_k (g_L \sin\theta_{hk} + b_L \cos\theta_{hk}) \ ,
\end{aligned}
$$

where v_i, $i = h, k$, is the voltage at bus i; p_i, q_i the active and reactive power injection at bus i; θ_i is the voltage angle at bus i; $\theta_{hk} = \theta_h - \theta_k$; $g_L + jb_L = (r_L + jx_L)^{-1}$, where $r_L = 0.01$ pu, $x_L = 0.15$ pu (Ω) the series resistance and reactance, respectively.

- Constant voltage model equations:

$$
\begin{aligned}
0 &= v_{k,0} - v_k \ , \\
0 &= \theta_{k,0} - \theta_k \ ,
\end{aligned}
$$

where $v_{k,0} = 1.01$ pu (kV) the initial voltage at bus k, $\theta_{k,0} = 0^\circ$ the initial voltage angle at bus k.

- AVR equations:

$$
\begin{aligned}
T_a \dot{v}_{r1} &= K_a(v_{ref} - u_m - u_{r2} - \tilde{v}_f K_f / T_f) - v_{r1} \ , \\
T_f \dot{v}_{r2} &= -v_f K_f / T_f + v_{r2} \ , \\
T_e \dot{v}_f &= -K_e v_f - v_{r1} \ , \\
T_r \dot{v}_m &= u_h - u_m \ ,
\end{aligned}
$$

where v_{r1}, v_{r2}, v_m are state variables and $v_{ref} = 1.1$ pu (kV) is the voltage reference; $T_r = 0.002$ s the measurement time constant; $K_e = 0.5$, $T_e = 1$ s the field circuit integral deviation and time constant; $K_a = 10$, $T_a = 0.2$ s are the amplifier gain and time constant; $K_f = 0.09$, $T_f = 0.3$ s the stabilizer gain and time constant, respectively.

$$
A = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -0.39 & 0.44 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.66 & -0.79 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1.4 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1.2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.5 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -10 & 0 & -1 & -10 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 6.92 & -6.92 & 1.04 & 0.13 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -6.84 & 6.84 & -1.02 & -0.14 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.13 & -0.13 & 6.9 & -6.85 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1.05 & -1.05 & -6.64 & 6.58 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0.79 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.79 & 0 & 0.64 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
-0.66 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.66 & 0 & 0.77 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.3 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.5 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.44 & 0.39 & 0.66 & 0.79 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.39 & 0.44 & 0.79 & -0.66 & 0 & -1
\end{bmatrix},
$$

Box I.

The linearized system matrices A, E are as follows (see A in Box I):

$$
E = \begin{bmatrix}
0.003 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0.6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0.002 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.09 & 0 & 0.3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}.
$$

The pencil $sE - A$ has 8 finite eigenvalues $s_{1,2} = -0.238 \pm 0.335i$, $s_{3,4} = -0.68 \pm 11i$, $s_5 = -3.689$, $s_{6,7} = -4.281 \pm 3.884i$, $s_8 = -500$ and infinite eigenvalues $s_9, s_{10}, \ldots, s_{23}$. Since the algebraic multiplicity of all finite eigenvalues

$$
Q_q = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-0.23 & -1.0 & -0.05 & 0 & 0 & -0.01 & 0 & 0 & -0.05 & -0.01 & 0.08 & -0.14 & -0.01 & -0.01 & -0.05 \\
-0.28 & 0.13 & -1.0 & 0 & 0 & 0 & 0 & 0 & 0.02 & -0.02 & 0.09 & -0.02 & -0.09 & -0.06 & 0.02 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0.14 & -0.06 & -0.09 & 1.0 & 0 & 0 & 0 & 0 & -0.02 & -0.02 & 0.11 & -0.12 & -0.06 & -0.05 & -0.03 \\
0 & 0 & 0 & 0 & 1.0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1.0 & 0 & 0 & -0.05 & 0 & 0.02 & -0.03 & 0.06 & 0.04 & -0.11 \\
0 & 0 & 0 & 0 & 0 & 0 & -1.0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.02 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1.0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0.14 & -0.06 & -0.08 & -0.01 & 0 & 0 & 0 & 0 & 1.0 & -0.03 & 0.01 & 0.1 & -0.46 & -0.32 & 0.73 \\
-0.99 & 0.44 & 0.59 & 0.04 & 0 & 0.03 & 0 & 0 & 0.24 & 1.0 & -0.74 & 0.88 & 0.34 & 0.27 & 0.34 \\
0.28 & -0.13 & -0.17 & -0.01 & 0 & -0.01 & 0 & 0 & -0.07 & 0.09 & -1.0 & 0.07 & 0.08 & 0.06 & -0.03 \\
-0.23 & 0.1 & 0.14 & 0.01 & 0 & 0.01 & 0 & 0 & 0.06 & -0.07 & -0.2 & -0.99 & 0.01 & 0 & -0.11 \\
0.78 & -0.35 & -0.47 & -0.03 & 0 & -0.02 & 0 & 0 & -0.19 & 0.24 & 0.67 & -0.78 & 0.96 & 0.04 & -0.73 \\
0.57 & -0.25 & -0.34 & -0.02 & 0 & -0.02 & 0 & 0 & -0.14 & 0.17 & 0.49 & -0.57 & -0.57 & -1.0 & 0.27 \\
1.0 & -0.44 & -0.6 & -0.04 & 0 & -0.03 & 0 & 0 & -0.24 & 0.3 & 0.85 & -1.0 & -1.0 & 0.07 & -0.36 \\
0.03 & -0.01 & -0.02 & 0 & 0 & 0 & 0 & 0 & -0.01 & 0.01 & 0.03 & -0.03 & -0.03 & 0 & -1.0
\end{bmatrix},
$$

Box II.

is 1, we have $p_* = 1$. Regarding the matrices P, Q, we have

$$
P = \begin{bmatrix} P_p \\ P_q \end{bmatrix}, Q = \begin{bmatrix} Q_p & Q_q \end{bmatrix},
$$

where (see Q_q in Box II and P_q^T in Box III)

$$
Q_p = \begin{bmatrix}
0 & -0.5+0.05i & -0.5-0.05i & 0 & 0 & 0.44 & -0.14-0.28i & -0.14+0.28i \\
0 & -0.02i & 0.02i & 0 & 0 & -0.01 & 0 & 0 \\
0 & -0.01i & 0.01i & 0 & 0 & 0.04 & 0.12+0.25i & 0.12-0.25i \\
0 & -0.02+0.09i & -0.02-0.09i & 0 & 0 & 0.94 & -0.08-0.13i & -0.08+0.13i \\
0 & 0.23-0.03i & 0.23+0.03i & 0 & 0 & -0.2 & 0.11+0.22i & 0.11-0.22i \\
0 & 0.29-0.05i & 0.29+0.05i & 0 & 0 & -0.47 & 0.1+0.18i & 0.1-0.18i \\
-1.0 & -0.01+0.01i & -0.01-0.01i & 0 & 0 & 0.15 & 0.02+0.05i & 0.02-0.05i \\
0 & 0 & 0 & 0.13+0.13i & 0.13-0.13i & -0.05 & -0.73+0.27i & -0.73-0.27i \\
-0.1 & -0.03-0.04i & -0.03+0.04i & -1.0 & -1.0 & 0.15 & -0.28-0.17i & -0.28+0.17i \\
0 & 0 & 0 & 0.01+0.08i & 0.01-0.08i & -0.15 & -0.03i & 0.03i \\
0 & -0.14-0.01i & -0.14+0.01i & 0 & 0 & -0.04 & -0.01i & 0.01i \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -0.01+0.01i & -0.01-0.01i & 0 & 0 & 0.14 & 0.02+0.05i & 0.02-0.05i \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -0.11-0.07i & -0.11+0.07i & 0.01i & -0.01i & -1.0 & -0.14-0.33i & -0.14+0.33i \\
0 & 0.96+0.03i & 0.96-0.03i & 0 & 0 & 0.09 & 0 & 0 \\
0 & -0.29+0.05i & -0.29-0.05i & 0 & 0 & 0.47 & -0.1-0.18i & -0.1+0.18i \\
0 & 0.23-0.03i & 0.23+0.03i & 0 & 0 & -0.2 & 0.11+0.22i & 0.11-0.22i \\
0 & -0.78+0.05i & -0.78-0.05i & 0 & 0 & 0.81 & 0.05+0.12i & 0.05-0.12i \\
0 & -0.53-0.09i & -0.53+0.09i & 0 & 0 & -0.95 & -0.04-0.11i & -0.04+0.11i \\
0 & -0.97-0.03i & -0.97+0.03i & 0 & 0 & -0.09 & 0 & 0 \\
0 & -0.06+0.07i & -0.06-0.07i & -0.01i & 0.01i & 1.0 & 0.15+0.33i & 0.15-0.33i
\end{bmatrix},
$$

$$
P_q^T = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0.07 & -1.0 & 0 & 0 & 0 & 0.01 & 0 & 0 & 0 & -1.0 & 0.22 & 0.16 & 0.71 & -0.5 \\
0 & 1.0 & 0.01 & 0 & 0 & 0 & 0.02 & 0 & 0 & 0 & 0.01 & -0.92 & -0.48 & -0.08 & -0.93 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1.0 & 0 & -0.07 & 0 & 0 & 0.09 & -0.99 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -0.99 & 0 & 0.06 & 0.01 & 0 & -0.09 & -1.0 & 0 & 0 & 0 & 0 & 0 \\
0 & -0.01 & 0.03 & 0.02 & 0 & -1.0 & 0.16 & 0 & -0.96 & 0 & -0.1 & -0.08 & -0.06 & 0.02 & 0.15 \\
0 & 0.01 & 0.01 & 0.15 & 0 & 0.98 & 0.03 & 0 & -1.0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1.0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0.11 & -0.06 & 0 & 0 & 0 & 1.0 & 0 & 0.04 & 0 & 0.67 & 0.58 & 0.4 & -0.11 & -1.0 \\
0 & 0 & 0 & 0 & 1.0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1.0 & 0.07 & -0.04 & 0 & 0 & -0.09 & -0.59 & 0 & -0.11 & 0 & 0.43 & 0.37 & 0.26 & -0.07 & -0.64 \\
0.83 & 0.09 & -0.04 & 0 & 0 & -0.11 & -0.71 & 0 & -0.13 & 0 & 0.52 & 0.45 & 0.31 & -0.09 & -0.77 \\
0 & -0.87 & 0.01 & 0 & 0 & 0 & 0.02 & 0 & 0 & 0 & 0.01 & -0.92 & -0.48 & -0.08 & -0.93 \\
0 & -0.05 & -0.96 & 0 & 0 & 0 & -0.01 & 0 & 0 & 0 & 1.0 & -0.22 & -0.16 & -0.71 & 0.5 \\
0 & -0.01 & 0.03 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.9 & -0.53 & 1.0 & -1.0 & -0.01 \\
0 & -0.03 & 0.04 & 0 & 0 & 0 & -0.01 & 0 & 0 & 0 & 0.76 & -1.0 & 0.9 & 0.86 & 0.36
\end{bmatrix}.
$$

Box III.

$$
P_p^T = \begin{bmatrix}
0 & -0.98+0.02i & -0.98-0.02i & 0.63-0.37i & 0.63+0.37i & -1.0 & -0.21+0.03i & -0.21-0.03i \\
0 & -0.02i & 0.02i & -0.01+0.02i & -0.01-0.02i & 0.05 & 0.05-0.09i & 0.05+0.09i \\
0 & 0 & 0 & 0 & 0 & 0 & 0.06i & -0.06i \\
0 & 0 & 0 & -0.03+0.04i & -0.03-0.04i & 0.09 & 0.07-0.04i & 0.07+0.04i \\
0.14 & 0.03i & -0.03i & -0.12-0.02i & -0.12+0.02i & 0.09 & -0.07+0.08i & -0.07-0.08i \\
0.27 & -0.02i & 0.02i & -0.18-0.06i & -0.18+0.06i & 0.07 & -0.18+0.09i & -0.18-0.09i \\
-1.0 & 0 & 0 & 0.59+0.35i & 0.59-0.35i & 0 & 0.89-0.05i & 0.89+0.05i \\
0 & 0 & 0 & -0.04+0.04i & -0.04-0.04i & 0 & -0.08+0.01i & -0.08-0.01i \\
0 & 0 & 0 & -0.06-0.03i & -0.06+0.03i & 0 & -0.09 & -0.09 \\
0 & 0 & 0 & -0.4+0.41i & -0.4-0.41i & 0.01 & 0.95+0.05i & 0.95-0.05i \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0.1 & 0 & 0 & -0.06-0.04i & -0.06+0.04i & -0.02 & -0.1-0.01i & -0.1+0.01i \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-0.71 & 0 & 0 & 0.38+0.3i & 0.38-0.3i & 0.11 & 0.71+0.08i & 0.71-0.08i \\
0 & 0.03i & -0.03i & -0.01 & -0.01 & 0.01 & 0 & 0 \\
0.18 & 0.02i & -0.02i & -0.14-0.04i & -0.14+0.04i & 0.08 & -0.11+0.08i & -0.11-0.08i \\
0.22 & -0.02i & 0.02i & -0.15-0.04i & -0.15+0.04i & 0.08 & -0.14+0.1i & -0.14-0.1i \\
0.27 & -0.03i & 0.03i & -0.18-0.05i & -0.18+0.05i & 0.1 & -0.16+0.06i & -0.16-0.06i \\
-0.14 & -0.02i & 0.02i & 0.13+0.01i & 0.13-0.01i & -0.11 & 0.05-0.04i & 0.05+0.04i \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-0.1 & 0 & 0 & 0.06+0.04i & 0.06-0.04i & 0.02 & 0.1+0.01i & 0.1-0.01i
\end{bmatrix},
$$

Table 1
Values of $C(s_j)$ for $K_a = 10.1$.

Eigenvalue	$C(s_j)$
$s_{1,2} = -0.238 \pm 0.335i$	0.6503
$s_{3,4} = -0.68 \pm 11i$	0.9898
$s_5 = -3.689$	0.5743
$s_{6,7} = -4.281 \pm 3.884i$	0.7764
$s_8 = -500$	0.9968

Table 2
Values of $C(s_j)$ for $K_a = 11$.

Eigenvalue	$C(s_j)$
$s_{1,2} = -0.238 \pm 0.335i$	0.6577
$s_{3,4} = -0.68 \pm 11i$	0.9972
$s_5 = -3.689$	0.5817
$s_{6,7} = -4.281 \pm 3.884i$	0.7838
$s_8 = -500$	1.0042

We now consider a perturbation of the AVR amplifier gain. The perturbed gain is $K_a = 10+10\lambda$. Then, the perturbation matrix ΔA is:

$$\Delta A = \begin{bmatrix}
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&-10p&0&0&-10p&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0
\end{bmatrix}.$$

4.2.1. Case 1

For $\lambda = 0.01$, $K_a = 10.1$, the values of $C(s_j)$ for the finite eigenvalues of the system are summarized in Table 1. As it can be seen $C(s_j) < 1$ for all finite eigenvalues and therefore, according to Theorem 3.1, the perturbed system is stable. Indeed, eigenvalue analysis shows that the rightmost eigenvalues of the perturbed system are $\hat{s}_{1,2} = -0.237 \pm 0.336i$.

4.2.2. Case 2

For $p = 0.1$, $K_a = 11$, the values of $C(s_j)$ for the finite eigenvalues of the system are summarized in Table 2. From this table we notice that $C(s_8) > 1$, which indicates that Theorem 3.1 is inconclusive for the stability of the perturbed system for $\lambda = 0.1$. Eigenvalue analysis of the perturbed system shows that the rightmost eigenvalues are $\hat{s}_{1,2} = -0.233 \pm 0.344i$, which indicate that this system is stable.

In electric power engineering, the study of the stability properties of a power system around an operating point is known as small-signal stability analysis (SSSA). SSSA indicates if the system is stable when subjected to “small” disturbances. In this context, the condition obtained in Theorem 3.1 allows identifying whether a given disturbance is sufficiently small so that the system linearization can be considered to remain valid.

Conclusions

In this article we provide a practical test for robust stability for a class of singular linear systems of differential equations. This test requires only the knowledge of the invariants of the initial system and can be used without resorting to any further processes of computations to obtain invariants of any other perturbed system. Two numerical examples provide further insight. The second case study, in particular, discusses a practical application of the theoretical results provided in this paper. A further extension of this work is to apply this method into singular linear systems of fractional differential equations. The concept of stability of fractional differential equations is different than that of the systems considered in this article and we believe that our proposed method can provide interesting results also for the robustness of this type of systems. Finally, we will dedicate future work to study the compatibility of our approach with sparse matrix numerical methods.

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