The Möbius transform effect in singular systems of differential equations

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Abstract: The main objective of this article is to provide a link between the solutions of an initial value problem of a linear singular system of differential equations and the solutions of its proper M-systems, i.e. systems that appear after applying the generalized Möbius transform to the pencil of the original singular system (prime system). Firstly, we prove that by using the pencil of the prime system we can study the existence and uniqueness of solutions of its proper M-systems. Moreover these solutions can be explicitly represented without resorting to any further processes of computations. Finally, numerical examples are given to illustrate our theory.

Keywords : singularity, dynamical system, Möbius transform, initial conditions, differential equations

1 Introduction

Singular systems of linear differential/difference equations are inherent in many physical, engineering, mechanical, and financial models. Having in mind such applications, for instance in finance, we mention the well-known input-output Leontief model and its several important extensions, see [3], [6], [8]. Another application of a singular system is the constrained mechanical and robotic system described in [15]. Singular systems also appear in control theory, see [5], in macroeconomics, see [7], circuit theory, see [18], and in the modeling of power systems, see [20], [21], [23]. For some other recent contributions on singular systems see [1], [4], [10], [11], [12], [19], [25], [26], [28], and the references therein.

We consider the following system:

$$EY'(t) = AY(t), \tag{1}$$

where $E, A \in \mathbb{C}^{r\times m}$, $Y : [0, +\infty] \to \mathbb{C}^{m\times 1}$. The matrices E, A can be non-square ($r \neq m$), or square ($r = m$) with E singular ($\det E$=0). By Y' we denote the first order derivative of $Y(t)$. The pencil $sE - A$ is then used to study this system. A matrix pencil is a family of matrices $sE - A$, parametrized by a complex number s, see [9], [16]:

Definition 1.1. Given E, $A \in \mathbb{C}^{r\times m}$, and an arbitrary $s \in \mathbb{C}$, the matrix pencil $sE - A$ is called:

1. Regular when $r = m$ and $\det(sE - A) \not\equiv 0$;
2. Singular when $r \neq m$ or $r = m$ and $\det(sE - A) \equiv 0$.

In this article we consider the case that the pencil $sE - A$ is *regular* with E *singular*. This type of pencil, see [2], [16], [27], has finite eigenvalues which are the zeros of the function $\det(sE - A)$, and eigenvalues that tend to infinity. The existence of an infinite eigenvalue in pencils of singular systems can be seen if we write the generalized eigenvalue problem in the reciprocal form $EX = s^{-1}AX$. If E is singular with a null vector X, then $EX = 0_{m,1}$, so that X is an eigenvector of the reciprocal problem corresponding to the eigenvalue $s^{-1} = 0$, i.e. $s \to \infty$.

Definition 1.2. The general form of the Möbius transformation, or linear fractional transformation, is given by

$$s := f(z) = \frac{az+b}{cz+d}, \quad a, b, c, d \in \mathbb{C}, \quad ad - bc \neq 0. \tag{2}$$

The restriction in this definition is necessary because if $ad = bc$ then s is constant which can not be possible. We consider the pencil $sE - A$ of system (1). Its eigenvalues are then given by solving the following characteristic equation:

$$\left|sE - A\right| = 0,$$

whereby applying the transform (2) we get

$$\left|\frac{az+b}{cz+d}E - A\right| = 0,$$

or, equivalently, by using determinant properties

$$\left|(az+b)E - (cz+d)A\right| = 0,$$

or, equivalently,

$$\left|(aE - cA)z - (dA - bE)\right| = 0,$$

which is the characteristic equation of a linear dynamical system

$$\tilde{E}\tilde{Y}'(t) = \tilde{A}\tilde{Y}(t), \tag{3}$$

with pencil

$$z\tilde{E} - \tilde{A},$$

where

$$\tilde{E} = aE - cA, \quad \tilde{A} = dA - bE.$$

Definition 1.3. The system (1) will be referred to as the prime system, and the family of systems (3) will be defined as its proper $M(a, b, c, d)$-systems, or simply M-systems.

The importance of the family of systems of type (3) has been further emphasized by their role on specific cases in defining some important notions as duality, or studying the stability of a discrete time system through the spectrum of the pencil of (1). This can be seen as follows. If we consider the transformation (2) for $a = d = 0$, and $b = c = 1$, then

$$s = \frac{1}{z},$$

and by applying it to the pencil of (1) we get the pencil

$$Az - E,$$

which is the pencil of the dual system

$$AY'(t) = EY(t),$$

of system (1). Some studies on the duality of systems can be found in [7], [17], [22].

Furthermore, if we consider the inequality $Re(s) < 0$, and rewrite it as

$$\frac{s + \bar{s}}{2} < 0,$$

or, equivalently,

$$s + \bar{s} < 0,$$

by applying (2) for $a = c = d = 1$, and $b = -1$, we get

$$\frac{z - 1}{z + 1} + \frac{\bar{z} - 1}{\bar{z} + 1} < 0,$$

or, equivalently,

$$(\bar{z} + 1)(z - 1) + (\bar{z} - 1)(z + 1) < 0,$$

or, equivalently, by taking into account that $\bar{z}z = |z|^2$

$$|z| < 1.$$

Hence the set $\{Re(w) < 0, \quad \forall w \in \mathbb{C}\}$ maps to the set $\{|z| < 1, \quad \forall z \in \mathbb{C}\}$ through (1) for $a = c = d = 1$, and $b = -1$, i.e.

$$s = \frac{z - 1}{z + 1}.$$

Thus if we consider the transformation (2), by applying it to the pencil of system (1), the stability of this continuous time system can be studied through the stability of the discrete time system

$$(E - A)X_{k+1} = (A + E)X_k,$$

with pencil

$$(E - A)z - (A + E),$$

where $X : \mathbb{N} \to \mathbb{C}^{m\times 1}$. For example if we consider system (1) for $E = I_m$, i.e.

$$Y' = AY,$$

then instead of studying the eigenvalues of A with $Re(w) < 0$ we can study the eigenvalues of the pencil $(I - A)z - (I + A)$ for $|z| < 1$.

The corresponding matrix pencil of (1) is $sE - A$, and of (3) $z\tilde{E} - \tilde{A}$. It is clear that the essence of the above type depends on the relationships between the associated pencils. The study of the relationship between (1), (3) is reduced to an investigation of the links between their pencils. This notion defined above may be qualified algebraically in terms of relationships between the strict equivalence invariants of the associated pencils. A main result of this article is that, if the solution of the prime system is known, then the solution of any of its M-systems can be represented without further computation.

The paper is organized as follows: in section 2 we refer to the mathematical background used throughout this paper, in section 3 we provide properties of existence and uniqueness of solutions for the proper M-systems of (1) by only using the invariants & properties of the pencil of (1). Furthermore, an explicit formula of solutions is given in the case that there exists a unique solution. Finally, in section 4, we provide numerical examples to illustrate our theory.

Throughout the paper, by 0_{ij} we will denote the zero matrix of i rows and j columns, by ^{T} the transpose tensor, and by I_m the identity matrix $m \times m$. Finally, let $B_{n_1} \in \mathbb{C}^{n_1\times n_1}$,$B_{n_2} \in \mathbb{C}^{n_2\times n_2}$, ..., $B_{n_r} \in \mathbb{C}^{n_r\times n_r}$. By the direct sum

$$B_{n_1} \oplus B_{n_2} \oplus \cdots \oplus B_{n_r}$$

we will denote the block diagonal matrix:

$$blockdiag \begin{bmatrix} B_{n_1} & B_{n_1} & \dots & B_{n_r} \end{bmatrix}.$$

2 Mathematical background and notation

This section introduces some preliminary concepts and definitions from matrix pencil theory, which are used throughout the paper. The connection between (1), (3), or between their matrix pencils $sE - A$, $z\tilde{E} - \tilde{A}$, may be seen as a consequence of the special transformation (2). The notions of their relation may be qualified algebraically in terms of relationships between the strict equivalence invariants of the associated pencils. These relationships are summarized bellow. Let s_i, z_i, $i = 1, 2, ..., \nu$ be non-zero finite eigenvalue:

1. For $a, c, d \neq 0$:
   - If $s \longrightarrow 0$ then $z \longrightarrow -\frac{b}{a}$;
   - If $s \longrightarrow \infty$ then $z \longrightarrow -\frac{d}{c}$;
   - If $s \longrightarrow s_i$ then $z \longrightarrow \frac{-ds_i+b}{cs_i-a}$;
   - If $z \longrightarrow 0$ then $s \longrightarrow \frac{b}{d}$;

- If $z \longrightarrow \infty$ then $s \longrightarrow -\frac{a}{c}$;
- If $z \longrightarrow z_i$ then $s \longrightarrow \frac{az_i+b}{cz_i+d}$;

2. For $a = 0$ then:
   - If $s \longrightarrow 0$ then $z \longrightarrow \infty$;
   - If $s \longrightarrow \infty$ then $z \longrightarrow -\frac{d}{c}$;
   - If $s \longrightarrow s_i$ then $z \longrightarrow \frac{-ds_i+b}{cs_i}$;
   - If $z \longrightarrow 0$ then $s \longrightarrow \frac{b}{d}$;
   - If $z \longrightarrow \infty$ then $s \longrightarrow 0$;
   - If $z \longrightarrow z_i$, s_i, $i = 1, 2, ..., \nu$ then $s \longrightarrow \frac{b}{cz_i+d}$;
3. For $c = 0$:
   - If $s \longrightarrow 0$ then $z \longrightarrow -\frac{b}{a}$;
   - If $s \longrightarrow \infty$ then $z \longrightarrow \infty$;
   - If $s \longrightarrow s_i$, s_i then $z \longrightarrow \frac{-ds_i+b}{-a}$;
   - If $z \longrightarrow 0$ then $s \longrightarrow \frac{b}{d}$;
   - If $z \longrightarrow \infty$ then $s \longrightarrow \infty$;
   - If $z \longrightarrow z_i$ then $s \longrightarrow \frac{az_i+b}{d}$;
4. For $d = 0$:
   - If $s \longrightarrow 0$ then $z \longrightarrow -\frac{b}{a}$;
   - If $s \longrightarrow \infty$ then $z \longrightarrow 0$;
   - If $s \longrightarrow s_i$ then $z \longrightarrow \frac{b}{cs_i-a}$;
   - If $z \longrightarrow 0$ then $s \longrightarrow \infty$;
   - If $z \longrightarrow \infty$ then $s \longrightarrow -\frac{a}{c}$;
   - If $z \longrightarrow z_i$ then $s \longrightarrow \frac{az_i+b}{cz_i}$.

As already mentioned in the previous section, in this article we consider that the pencil is *regular* with invariants of the following type:

- ν finite eigenvalues of algebraic multiplicity p_i, $i = 1, 2, ..., \nu$;
- an infinite eigenvalue of algebraic multiplicity q,

where $\sum_{i=1}^{\nu} p_i = p$ and $p + q = m$. Unlike singular pencils, when a system has a regular one there always exist solutions. This can be seen as follows. Let $\mathcal{L}\{Y(t)\} = Z(s)$ be the Laplace transform of $Y(t)$. By applying the Laplace transform $\mathcal{L}$ to (1) we get

$$E\mathcal{L}\{Y'(t)\} = A\mathcal{L}\{Y(t)\},$$

or, equivalently,

$$E(sZ(s) - Y_0) = AZ(s),$$

where $Y_0 = Y(0)$, i.e. the initial condition of (1). Since we assume that Y_0 is unknown we can use an unknown constant vector $C \in \mathbb{C}^{m\times 1}$ and give to the above expression the following form

$$(sE - A)Z(s) = EC.$$

From the above expression it can be seen that when $sE - A$ is a regular pencil, i.e. $\det(sE - A) \not\equiv 0$, there always exists a solution while if the pencil is singular existence is not guaranteed. However, even if there exist solutions for a regular pencil, it is not guaranteed that for given initial conditions a singular system will have a unique solution, see also [5]. If the given initial conditions are consistent, and there exist solutions for (1), then in the formulas of the general solutions we replace $C = Y_0$. However, if the given initial conditions are non-consistent but there exist solutions for (1), then the general solution holds for $t > 0$ and not for $t = 0$.

If $sE - A$ is regular, then from its regularity there exist non-singular matrices P, $Q \in \mathbb{C}^{m\times m}$ such that

$$\begin{aligned} PEQ &= I_p \oplus H_q, \\ PAQ &= J_p \oplus I_q, \end{aligned} \tag{4}$$

where $J_p \in \mathbb{C}^{p\times p}$ is the Jordan matrix related to the finite eigenvalues, see [16], $H_q \in \mathbb{C}^{q\times q}$ is a nilpotent matrix with index q_*, constructed by using the algebraic multiplicity of the infinite eigenvalue, and $p + q = m$. The matrices P, Q contain all linearly independent left and right eigenvectors, respectively. Note that in singular systems, for a pencil, there may not always exist a full set of linearly independent eigenvectors that form a complete basis. This happens when the algebraic multiplicity of at least one eigenvalue is greater than its geometric multiplicity (the nullity of the matrix, or the dimension of its nullspace). In such cases, a generalized eigenvector of the pencil is a nonzero vector, which is associated with the eigenvalue s having algebraic multiplicity $k \geq 1$, satisfying $(sE - A)^k = 0_{m,1}$.

Proposition 2.1. Consider the system (1) with known initial conditions $Y(0) = Y_0$ and a regular pencil. Let J_p be the Jordan matrix of the finite eigenvalues, and Q_p the matrix that contains all linear independent eigenvectors. Then there exists a unique solution if and only if:

$$Y_0 \in \mathrm{colspanQ_p}.$$

In this case, the unique solution is given by

$$Y(t) = Q_p e^{J_p t} Z_p(0),$$

where $Z_p(0)$ is the unique solution of the linear system

$$Y_0 = Q_p Z_p(0).$$

Proof. By substituting the transformation

$$Y(t) = QZ(t).$$

into (1), and by multiplying by P we obtain

$$PEQZ'(t) = PAQZ(t).$$

Let Q_p, Q_q be the matrices that contain all eigenvectors of the finite, and infinite eigenvalues respectively. Then by setting

$$Z(t) = \begin{bmatrix} Z_p(t) \\ Z_q(t) \end{bmatrix}, \quad Q = \begin{bmatrix} Q_p & Q_q \end{bmatrix},$$

with $Z_{p_0}(t) \in \mathbb{C}^{p_0 \times 1}$, $Z_p(t) \in \mathbb{C}^{p \times 1}$, $Z_q(t) \in \mathbb{C}^{q \times 1}$, we arrive easily at two subsystems of (1):

$$Z_p'(t) = J_p Z_p(t);$$

$$H_q Z_q'(t) = Z_q(t).$$

The first subsystem has solution:

$$Z_p(t) = e^{J_p t} Z_p(0),$$

For the second subsystem let q_* be the index of the nilpotent matrix H_q, i.e. $H_q^{q_*} = 0_{q,q}$. Then if we obtain the following matrix equations

$$\begin{array}{c} H_q Z_q^{'}(t) = Z_q(t) \\ H_q^2 Z_q^{''}(t) = H_q Z_q^{'}(t) \\ H_q^3 Z_q^{'''}(t) = H_q^2 Z_q^{''}(t) \\ H_q^4 Z_q^{(4)}(t) = H_q^3 Z_q^{'''}(t) \\ \vdots \\ H_q^{q_*-1} Z_q^{(q_*-1)}(t) = H_q^{q_*-2} Z_q^{(q_*-2)}(t) \\ H_q^{q_*} Z_q^{(q_*}(t) = H_q^{q_*-1} Z_q^{(q_*-1)}(t) \end{array},$$

by taking the sum of the above equations we arrive easily at the solution:

$$Z_q(t) = 0_{q,1}.$$

By using the solutions of the two subsystems, we obtain:

$$Y(t) = QZ(t) = \begin{bmatrix} Q_p & Q_q \end{bmatrix} \begin{bmatrix} e^{J_p t} Z_p(0) \\ 0_{q,1} \end{bmatrix},$$

or, equivalently,

$$Y(t) = Q_p e^{J_p t} Z_p(0)$$

This solution is unique if and only if

$$Y_0 = Q_p Z_p(0),$$

or, equivalently,

$$Y_0 \in \mathrm{colspanQ_p}.$$

The proof is completed.

3 Main results

In this section we present our main results. By using only the invariants of the pencil of system (1), we will provide insight on the solutions of the family of systems (3). We provide the following Theorem:

Theorem 3.1. Consider system (1) with a regular pencil, and the family of systems (3) with known initial conditions $\tilde{Y}(0) = \tilde{Y}_0$. Then:

(a) If $a, c \neq 0$, then there exists a unique solution for (3) if and only if:

$$\tilde{Y}_0 \in \text{colspan}\begin{bmatrix} Q_p & Q_q \end{bmatrix},$$

where:

- $Q_p \in \mathbb{C}^{m\times p}$ are the linear independent eigenvectors (including the generalized) of all finite eigenvalue of $sE - A$ except the eigenvectors of s_0, an eigenvalue of the pencil $sE - A$ such that $a = cs_0$;
- $Q_q \in \mathbb{C}^{m\times q}$ are the linear independent eigenvectors (including the generalized) of the infinite eigenvalue of $sE - A$.

Then the solution is given by:

$$\tilde{Y}(t) = \begin{bmatrix} Q_p & Q_q \end{bmatrix} e^{J_{p+q}t}\tilde{Z}_{p+q}(0),$$

where $e^{J_{p+q}t} = e^{\tilde{J}_p t} \oplus e^{\tilde{J}_q t}$, $\tilde{Z}_{p+q}(0) = \begin{bmatrix} \tilde{Z}_p(0) \\ \tilde{Z}_q(0) \end{bmatrix}$, $\tilde{J}_p = (aI_p - cJ_p)^{-1}(dJ_p - bI_p)$, $\tilde{J}_q = (aH_q - cI_q)^{-1}(dI_q - bH_q)$, $J_p \in \mathbb{C}^{p\times p}$ is the Jordan matrix related to the finite eigenvalues except s_0, and H_q is a nilpotent matrix with index q_*, constructed by using the algebraic multiplicity of the infinite eigenvalue. Finally, $\tilde{Z}_{p+q}(0)$ is the unique solution of the algebraic system

$$\tilde{Y}_0 = \begin{bmatrix} Q_p & Q_q \end{bmatrix} \tilde{Z}_{p+q}(0).$$

(b) If $a = 0$, then there exists a unique solution for (3) if and only if:

$$\tilde{Y}_0 \in \text{colspan}\begin{bmatrix} Q_p & Q_q \end{bmatrix},$$

where:

- $Q_p \in \mathbb{C}^{m\times p}$ are the linear independent eigenvectors (including the generalized) of all non-zero finite eigenvalue of $sE - A$;
- $Q_q \in \mathbb{C}^{m\times q}$ are the linear independent eigenvectors (including the generalized) of the infinite eigenvalue of $sE - A$.

Then the solution is given by:

$$\tilde{Y}(t) = \begin{bmatrix} Q_p & Q_q \end{bmatrix} e^{J_{p+q}t}\tilde{Z}_{p+q}(0),$$

where $e^{J_{p+q}t} = e^{\tilde{J}_p t} \oplus e^{\tilde{J}_q t}$, $\tilde{Z}_{p+q}(0) = \begin{bmatrix} \tilde{Z}_p(0) \\ \tilde{Z}_q(0) \end{bmatrix}$, $\tilde{J}_p = -\frac{1}{c} J_p^{-1}(dJ_p - bI_p)$, $\tilde{J}_q = -\frac{1}{c}(dI_q - bH_q)$, $J_p \in \mathbb{C}^{p \times p}$ is the Jordan matrix related to the finite non-zero eigenvalues, and H_q is a nilpotent matrix with index q_*, constructed by using the algebraic multiplicity of the infinite eigenvalue. Finally, $\tilde{Z}_{p+q}(0)$ is the unique solution of the algebraic system

$$\tilde{Y}_0 = \begin{bmatrix} Q_p & Q_q \end{bmatrix} \tilde{Z}_{p+q}(0).$$

(c) If $c = 0$, then there exists a unique solution for (3) if and only if:

$$\tilde{Y}_0 \in \mathrm{colspanQ_p},$$

where:

- $Q_p \in \mathbb{C}^{m \times p}$ are the linear independent eigenvectors (including the generalized) of all finite eigenvalue of $sE - A$;

Then the solution is given by:

$$\tilde{Y}(t) = Q_p e^{\tilde{J}_p t} \tilde{Z}_p(0),$$

where $\tilde{J}_p = \frac{1}{a}(dJ_p - bI_p)$, $J_p \in \mathbb{C}^{p \times p}$ is the Jordan matrix related to the finite eigenvalues, and $\tilde{Z}_p(0)$ is the unique solution of the algebraic system

$$\tilde{Y}_0 = Q_p \tilde{Z}_p(0).$$

Proof.

(a) Let s_0 be an eigenvalue of the pencil $sE - A$ such that $a = cs_0$. Then we can give Q, defined in (4), the following form

$$Q = \begin{bmatrix} Q_{s_0} & Q_p & Q_q \end{bmatrix},$$

where:

- $Q_{s_0} \in \mathbb{C}^{m \times p_0}$ are the linear independent eigenvectors (including the generalized) of the eigenvalue s_0 of $sE - A$;
- $Q_p \in \mathbb{C}^{m \times p}$ are the linear independent eigenvectors (including the generalized) of all finite eigenvalue of $sE - A$ except the eigenvectors of s_0;
- $Q_q \in \mathbb{C}^{m \times q}$ are the linear independent eigenvectors (including the generalized) of the infinite eigenvalue of $sE - A$.

In this case (4) will take the form

$$PEQ = I_{s_0} \oplus I_p \oplus H_q,$$

$$PEQ = J_{s_0} \oplus J_p \oplus I_q,$$

where $J_{s_0} \in \mathbb{C}^{s_0 \times s_0}$, $J_p \in \mathbb{C}^{p\times p}$ are the Jordan matrices related to the s_0, and the finite eigenvalues except s_0, respectively, $H_q \in \mathbb{C}^{q\times q}$ is a nilpotent matrix with index q_*, constructed by using the algebraic multiplicity of the infinite eigenvalue, and $s_0 + p + q = m$. We apply the transformation

$$\tilde{Y}(t) = Q\tilde{Z}(t).$$

to (3), and multiply by P:

$$P\tilde{E}Q\tilde{Z}'(t) = P\tilde{A}Q\tilde{Z}(t),$$

or, equivalently,

$$P(aE - cA)Q\tilde{Z}'(t) = P(dA - bE)Q\tilde{Z}(t),$$

or, equivalently,

$$[(aI_{s_0} - cJ_{s_0}) \oplus (aI_p - cJ_p) \oplus (aH_q - cI_q)]\tilde{Z}'(t) =$$

$$[(dJ_{s_0} - bI_{s_0}) \oplus (dJ_p - bI_p) \oplus (dI_q - bH_q)]\tilde{Z}(t),$$

whereby setting

$$\tilde{Z}(t) = \begin{bmatrix} \tilde{Z}_{s_0}(t) \\ \tilde{Z}_p(t) \\ \tilde{Z}_q(t) \end{bmatrix},$$

with $\tilde{Z}_{s_0}(t) \in \mathbb{C}^{s_0\times 1}$, $\tilde{Z}_p(t) \in \mathbb{C}^{p\times 1}$, $\tilde{Z}_q(t) \in \mathbb{C}^{q\times 1}$, and using the above written notations we arrive easily at three subsystems of (3):

$$(aI_{s_0} - cJ_{s_0})\tilde{Z}'_{s_0}(t) = (dJ_{s_0} - bI_{s_0})\tilde{Z}_{s_0}(t);$$

$$(aI_p - cJ_p)\tilde{Z}'_p(t) = (dJ_p - bI_p)\tilde{Z}_p(t);$$

$$(aH_q - cI_q)\tilde{Z}'_q(t) = (dI_q - bH_q)\tilde{Z}_q(t).$$

Note that the matrix $aI_{s_0} - cJ_{s_0}$ has only zeros in its diagonal because $a = cs_0$. Furthermore the matrices $aI_p - cJ_p$, $aH_q - cI_q$ are both invertible since all elements in their diagonal are non-zero. The solution of the first subsystem is

$$\tilde{Z}_{s_0}(t) = 0_{p_0,1}.$$

This can be easily proved similarly to the relevant part of the proof of Proposition 2.1. The two other subsystems have solutions:

$$\tilde{Z}_p(t) = e^{\tilde{J}_p t}\tilde{Z}_p(0), \quad \text{and} \quad \tilde{Z}_q(t) = e^{\tilde{J}_q t}\tilde{Z}_q(0),$$

respectively, where

$$\tilde{J}_p = (aI_p - cJ_p)^{-1}(dJ_p - bI_p), \quad \tilde{J}_q = (aH_q - cI_q)^{-1}(dI_q - bH_q).$$

By using the solutions of the three subsystems, and the notation for Q as written in the beginning of the proof we obtain:

$$\tilde{Y}(t) = Q\tilde{Z}(t) = \begin{bmatrix} Q_{s_0} & Q_p & Q_q \end{bmatrix} \begin{bmatrix} 0_{s_0,1} \\ e^{\tilde{J}_p t}\tilde{Z}_p(0) \\ e^{\tilde{J}_q t}\tilde{Z}_q(0) \end{bmatrix},$$

or, equivalently,

$$\tilde{Y}(t) = Q_p e^{\tilde{J}_p t}\tilde{Z}_p(0) + Q_q e^{\tilde{J}_q t}\tilde{Z}_q(0),$$

or, equivalently,

$$\tilde{Y}(t) = \begin{bmatrix} Q_p & Q_q \end{bmatrix} e^{J_{p+q} t}\tilde{Z}_{p+q}(0),$$

where $e^{J_{p+q}t} = e^{\tilde{J}_p t} \oplus e^{\tilde{J}_q t}$, $\tilde{Z}_{p+q}(0) = \begin{bmatrix} \tilde{Z}_p(0) \\ \tilde{Z}_q(0) \end{bmatrix}$. This solution is unique if and only if

$$\tilde{Y}_0 \in \operatorname{colspan} \begin{bmatrix} Q_p & Q_q \end{bmatrix}.$$

In this case $\tilde{Z}_{p+q}(0)$ is the unique solution of

$$\tilde{Y}_0 = \begin{bmatrix} Q_p & Q_q \end{bmatrix} \tilde{Z}_{p+q}(0).$$

(b) Let 0 be an eigenvalue of the pencil $sE - A$. Then we can give Q, defined in (4), the following form

$$Q = \begin{bmatrix} Q_{p_0} & Q_p & Q_q \end{bmatrix},$$

where:

- $Q_{p_0} \in \mathbb{C}^{m\times p_0}$ are the linear independent eigenvectors (including the generalized) of the zero eigenvalue of $sE - A$;
- $Q_p \in \mathbb{C}^{m\times p}$ are the linear independent eigenvectors (including the generalized) of all non-zero finite eigenvalue of $sE - A$;
- $Q_q \in \mathbb{C}^{m\times q}$ are the linear independent eigenvectors (including the generalized) of the infinite eigenvalue of $sE - A$.

In this case (4) will take the form

$$PEQ = I_{p_0} \oplus I_p \oplus H_q,$$

$$PEQ = J_{p_0} \oplus J_p \oplus I_q,$$

where $J_{p_0} \in \mathbb{C}^{p_0\times p_0}$, $J_p \in \mathbb{C}^{p\times p}$ are the Jordan matrices related to the zero, and the finite non-zero eigenvalues, respectively, $H_q \in \mathbb{C}^{q\times q}$ is a nilpotent matrix with index q_*, constructed by using the algebraic multiplicity of the infinite eigenvalue, and $p_0 + p + q = m$. We apply the transformation

$$\tilde{Y}(t) = Q\tilde{Z}(t).$$

into (3), and multiply by P:

$$P\tilde{E}Q\tilde{Z}'(t) = P\tilde{A}Q\tilde{Z}(t),$$

or, equivalently,

$$-P(cA)Q\tilde{Z}'(t) = P(dA - bE)Q\tilde{Z}(t),$$

or, equivalently,

$$-c[J_{p_0} \oplus J_p \oplus I_q]\tilde{Z}'(t) = [(dJ_{p_0} - bI_{p_0}) \oplus (dJ_p - bI_p) \oplus (dI_q - bH_q)]\tilde{Z}(t),$$

whereby setting

$$\tilde{Z}(t) = \begin{bmatrix} \tilde{Z}_{p_0}(t) \\ \tilde{Z}_p(t) \\ \tilde{Z}_q(t) \end{bmatrix},$$

with $\tilde{Z}_{p_0}(t) \in \mathbb{C}^{p_0 \times 1}$, $\tilde{Z}_p(t) \in \mathbb{C}^{p \times 1}$, $\tilde{Z}_q(t) \in \mathbb{C}^{q \times 1}$, and using the above written notations we arrive easily at three subsystems of (3):

$$cJ_{p_0}\tilde{Z}'_{p_0}(t) = (dJ_{p_0} - bI_{p_0})\tilde{Z}_{p_0}(t);$$

$$-cJ_p\tilde{Z}'_p(t) = (dJ_p - bI_p)\tilde{Z}_p(t);$$

$$-cI_q\tilde{Z}'_q(t) = (dI_q - bH_q)\tilde{Z}_q(t).$$

Note that $c \neq 0$ since $a = 0$ and the relation in (2) must hold. Furthermore, the matrix $-cJ_{p_0}$ has only zeros in its diagonal because J_{p_0} is the Jordan matrix of the zero eigenvalue of $sE - A$. Finally, the matrices $-cJ_p$, $-cI_q$ are both invertible since all elements in their diagonal are non-zero. The solution of the first subsystem is

$$\tilde{Z}_{p_0}(t) = 0_{p_0,1}.$$

This can be easily proved similarly to the relevant part of the proof of Proposition 2.1. The two other subsystems have solutions:

$$\tilde{Z}_p(t) = e^{\tilde{J}_p t}\tilde{Z}_p(0), \quad \text{and} \quad \tilde{Z}_q(t) = e^{\tilde{J}_q t}\tilde{Z}_q(0),$$

respectively, where

$$\tilde{J}_p = -\frac{1}{c}J_p^{-1}(dJ_p - bI_p), \quad \tilde{J}_q = -\frac{1}{c}(dI_q - bH_q).$$

By using the solutions of the three subsystems, and the notation for Q as written in the beginning of the proof we obtain:

$$\tilde{Y}(t) = Q\tilde{Z}(t) = \begin{bmatrix} Q_{s_0} & Q_p & Q_q \end{bmatrix} \begin{bmatrix} 0_{s_0,1} \\ e^{\tilde{J}_p t}\tilde{Z}_p(0) \\ e^{\tilde{J}_q t}\tilde{Z}_q(0) \end{bmatrix},$$

or, equivalently,

$$\tilde{Y}(t) = Q_p e^{\tilde{J}_p t}\tilde{Z}_p(0) + Q_q e^{\tilde{J}_q t}\tilde{Z}_q(0),$$

or, equivalently,

$$\tilde{Y}(t) = \begin{bmatrix} Q_p & Q_q \end{bmatrix} e^{J_{p+q}t}\tilde{Z}_{p+q}(0),$$

where $e^{J_{p+q}t} = e^{\tilde{J}_p t} \oplus e^{\tilde{J}_q t}$, $\tilde{Z}_{p+q}(0) = \begin{bmatrix} \tilde{Z}_p(0) \\ \tilde{Z}_q(0) \end{bmatrix}$. This solution is unique if and only if

$$\tilde{Y}_0 \in \text{colspan} \begin{bmatrix} Q_p & Q_q \end{bmatrix}.$$

In this case $\tilde{Z}_{p+q}(0)$ is the unique solution of

$$\tilde{Y}_0 = \begin{bmatrix} Q_p & Q_q \end{bmatrix} \tilde{Z}_{p+q}(0).$$

(c) We can give Q, defined in (4), the following form

$$Q = \begin{bmatrix} Q_p & Q_q \end{bmatrix},$$

where:

- $Q_p \in \mathbb{C}^{m \times p}$ are the linear independent eigenvectors (including the generalized) of all finite eigenvalue of $sE - A$;
- $Q_q \in \mathbb{C}^{m \times q}$ are the linear independent eigenvectors (including the generalized) of the infinite eigenvalue of $sE - A$.

From (4) we have:

$$PEQ = I_p \oplus H_q,$$

$$PEQ = J_p \oplus I_q,$$

where $J_p \in \mathbb{C}^{p \times p}$ is the Jordan matrix related to the finite eigenvalues, $H_q \in \mathbb{C}^{q \times q}$ is a nilpotent matrix with index q_*, constructed by using the algebraic multiplicity of the infinite eigenvalue, and $p + q = m$. We apply the transformation

$$\tilde{Y}(t) = Q\tilde{Z}(t).$$

into (3), and multiply by P:

$$P\tilde{E}Q\tilde{Z}'(t) = P\tilde{A}Q\tilde{Z}(t),$$

or, equivalently,

$$P(aE)Q\tilde{Z}'(t) = P(dA - bE)Q\tilde{Z}(t),$$

or, equivalently,

$$a[I_p \oplus H_q]\tilde{Z}'(t) = [(dJ_p - bI_p) \oplus (dI_q - bH_q)]\tilde{Z}(t),$$

whereby setting

$$\tilde{Z}(t) = \begin{bmatrix} \tilde{Z}_p(t) \\ \tilde{Z}_q(t) \end{bmatrix},$$

with $\tilde{Z}_p(t) \in \mathbb{C}^{p\times 1}$, $\tilde{Z}_q(t) \in \mathbb{C}^{q\times 1}$, and using the above written notations we arrive easily at two subsystems of (3):

$$aI_p\tilde{Z}_p'(t) = (dJ_p - bI_p)\tilde{Z}_p(t);$$

$$aH_q\tilde{Z}_q'(t) = (dI_q - bH_q)\tilde{Z}_q(t).$$

The solutions of the two subsystem are:

$$\tilde{Z}_p(t) = e^{\tilde{J}_p t}\tilde{Z}_p(0), \quad \text{and} \quad \tilde{Z}_q(t) = 0_{q,1},$$

respectively, where

$$\tilde{J}_p = \frac{1}{a}(dJ_p - bI_p).$$

By using the solutions of the two subsystems, and the notation for Q as written in the beginning of the proof we obtain:

$$\tilde{Y}(t) = Q\tilde{Z}(t) = \begin{bmatrix} Q_p & Q_q \end{bmatrix} \begin{bmatrix} e^{\tilde{J}_p t}\tilde{Z}_p(0) \\ 0_{q,1} \end{bmatrix},$$

or, equivalently,

$$\tilde{Y}(t) = Q_p e^{\tilde{J}_p t}\tilde{Z}_p(0).$$

This solution is unique if and only if

$$\tilde{Y}_0 \in \mathrm{colspanQ_p}.$$

In this case $\tilde{Z}_p(0)$ is the unique solution of

$$\tilde{Y}_0 = Q_p\tilde{Z}_p(0).$$

The proof is completed.

4 Numerical examples

In this Section we provide numerical examples to illustrate our theory.

Numerical example 1

We consider system (1) with

$$E = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \; A = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -4 & 2 & 2 & -3 & -2 & -1 \\ 1 & 1 & -1 & -1 & 0 & 0 \end{bmatrix}.$$

By applying the Möbius transform (2) to the pencil of (1) we arrive at the family of systems (3). Let $\tilde{Y}_0 = \begin{bmatrix} -4 & 6 & -5 & 7 & -7 & 9 \end{bmatrix}^T$, be the initial conditions of (3). We may now use Theorem 3.1, and the invariants of the pencil $sE-A$ of (1) in order to investigate the solutions of (3) $\forall a, b, c, d \in \mathbb{C}$. The pencil $sE-A$ has three finite eigenvalues $s_1 = 3$, $s_2 = 2$, $s_3 = 1$, of algebraic multiplicity $p_1 = p_2 = p_3 = 1$, and an infinite eigenvalue of algebraic multiplicity $q = 3$. The eigenspaces of $sE-A$ associated with the eigenvalues 3, 2, 1, are :

$$\left\langle u_1 \right\rangle = \left\langle \begin{bmatrix} -1 \\ 1 \\ -3 \\ 3 \\ -9 \\ 9 \end{bmatrix} \right\rangle, \quad \left\langle u_2 \right\rangle = \left\langle \begin{bmatrix} -1 \\ 1 \\ -2 \\ 2 \\ -4 \\ 4 \end{bmatrix} \right\rangle, \quad \left\langle u_3 \right\rangle = \left\langle \begin{bmatrix} -3 \\ 5 \\ -3 \\ 5 \\ -3 \\ 5 \end{bmatrix} \right\rangle,$$

while the eigenspace of $sE-A$ associated with the infinite eigenvalue, including the generalized eigenvectors, is:

$$\left\langle u_4, \quad u_5, \quad u_6 \right\rangle = \left\langle \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \\ 42 \\ -48 \end{bmatrix} \right\rangle.$$

The Jordan matrix related to the finite eigenvalues, and the matrix H_q are given by:

$$J_p = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \ H_q = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

The matrix Q_q is defined as $Q_q = \begin{bmatrix} u_4 & u_5 & u_6 \end{bmatrix}$. We have the following cases:

- If $a, c \neq 0$, then from Theorem 3.1, the general solution of system (3) is given by
$$\tilde{Y}(t) = \begin{bmatrix} Q_p & Q_q \end{bmatrix} e^{J_{p+q}t} \tilde{Z}_{p+q}(0).$$
The matrices $e^{J_{p+q}t}$, Q_p are defined as follows:

(i) If $a \neq 3c$, $a \neq 2c$, $a \neq c$, then $Q_p = \begin{bmatrix} u_1 & u_2 & u_3 \end{bmatrix}$, and:

$$e^{\tilde{J}_{p+q}t} = \begin{bmatrix} e^{\frac{-b+3d}{a-3c}t} & 0 & 0 & 0 & 0 & 0 \\ 0 & e^{\frac{-b+2d}{a-2c}t} & 0 & 0 & 0 & 0 \\ 0 & 0 & e^{\frac{-b+d}{a-c}t} & 0 & 0 & 0 \\ 0 & 0 & 0 & e^{-\frac{d}{c}t} & \frac{bc-ad}{c^2} & \frac{abc-ad}{c^3} \\ 0 & 0 & 0 & 0 & e^{-\frac{d}{c}t} & \frac{bc-ad}{c^2} \\ 0 & 0 & 0 & 0 & 0 & e^{-\frac{d}{c}t} \end{bmatrix}. \quad (5)$$

(ii) If $a = 3c$, $Q_p = \begin{bmatrix} u_2 & u_3 \end{bmatrix}$, and:

$$e^{\tilde{J}_{p+q}t} = \begin{bmatrix} e^{\frac{-b+2d}{a-2c}t} & 0 & 0 & 0 & 0 \\ 0 & e^{\frac{-b+d}{a-c}t} & 0 & 0 & 0 \\ 0 & 0 & e^{-\frac{d}{c}t} & \frac{bc-ad}{c^2} & \frac{abc-ad}{c^3} \\ 0 & 0 & 0 & e^{-\frac{d}{c}t} & \frac{bc-ad}{c^2} \\ 0 & 0 & 0 & 0 & e^{-\frac{d}{c}t} \end{bmatrix};$$

(iii) If $a = 2c$, then $Q_p = \begin{bmatrix} u_1 & u_3 \end{bmatrix}$, and:

$$e^{\tilde{J}_{p+q}t} = \begin{bmatrix} e^{\frac{-b+3d}{a-3c}t} & 0 & 0 & 0 & 0 \\ 0 & e^{\frac{-b+d}{a-c}t} & 0 & 0 & 0 \\ 0 & 0 & e^{-\frac{d}{c}t} & \frac{bc-ad}{c^2} & \frac{abc-ad}{c^3} \\ 0 & 0 & 0 & e^{-\frac{d}{c}t} & \frac{bc-ad}{c^2} \\ 0 & 0 & 0 & 0 & e^{-\frac{d}{c}t} \end{bmatrix};$$

(iv) If $a = c$, then $Q_p = \begin{bmatrix} u_2 & u_3 \end{bmatrix}$, and:

$$e^{\tilde{J}_{p+q}t} = \begin{bmatrix} e^{\frac{-b+3d}{a-3c}t} & 0 & 0 & 0 & 0 \\ 0 & e^{\frac{-b+2d}{a-2c}t} & 0 & 0 & 0 \\ 0 & 0 & e^{-\frac{d}{c}t} & \frac{bc-ad}{c^2} & \frac{abc-ad}{c^3} \\ 0 & 0 & 0 & e^{-\frac{d}{c}t} & \frac{bc-ad}{c^2} \\ 0 & 0 & 0 & 0 & e^{-\frac{d}{c}t} \end{bmatrix}.$$

It is easy to observe that in the case of (i), (ii), we have $\tilde{Y}_0 \in \text{colspan} \begin{bmatrix} Q_p & Q_q \end{bmatrix}$, and hence the solution is unique. In this case for both (i), (ii), the unique solution is given by:

$$\tilde{Y}(t) = \begin{bmatrix} -3e^{\frac{-b+d}{a-c}t} - e^{\frac{-b+2d}{a-2c}t} \\ 5e^{\frac{-b+d}{a-c}t} + e^{\frac{-b+2d}{a-2c}t} \\ -3e^{\frac{-b+d}{a-c}t} - 2e^{\frac{-b+2d}{a-2c}t} \\ 5e^{\frac{-b+d}{a-c}t} + 2e^{\frac{-b+2d}{a-2c}t} \\ -3e^{\frac{-b+d}{a-c}t} - 4e^{\frac{-b+2d}{a-2c}t} \\ 5e^{\frac{-b+d}{a-c}t} + 4e^{\frac{-b+2d}{a-2c}t} \end{bmatrix}. \tag{6}$$

It is also easy to observe that for both (iii), (iv), $\tilde{Y}_0 \notin \text{colspan} \begin{bmatrix} Q_p & Q_q \end{bmatrix}$, i.e. from Theorem 3.1 there does not exist a unique solution for these systems. Since $\tilde{Z}_{p+q}(0)$ can not be defined uniquely, we set $\tilde{Z}_{p+q}(0) = \begin{bmatrix} c_1 & c_3 & c_4 & c_5 & c_6 \end{bmatrix}$, and $\tilde{c}_4 = \frac{bc-ad}{c^2}c_5 + \frac{abc-ad}{c^3}c_6$, $\tilde{c}_5 = \frac{bc-ad}{c^2}c_6$. Then the general solution for (iii) is:

$$\tilde{Y}(t) = e^{\frac{-b+3d}{a-3c}t}c_1u_1 + e^{\frac{-b+d}{a-c}t}c_3u_3 + e^{\frac{-d}{c}t}\sum_{i=4}^{6} c_iu_i + \tilde{c}_4u_4 + \tilde{c}_5u_5.$$

For (iv) again $\tilde{Z}_{p+q}(0)$ can not be defined uniquely, we set $\tilde{Z}_{p+q}(0) = \begin{bmatrix} c_1 & c_2 & c_4 & c_5 & c_6 \end{bmatrix}$, and the general solution is:

$$\tilde{Y}(t) = e^{\frac{-b+3d}{a-3c}t} c_1 u_1 + e^{\frac{-b+2d}{a-2c}t} c_2 u_2 + e^{\frac{-d}{c}t} \sum_{i=4}^{6} c_i u_i + \tilde{c}_4 u_4 + \tilde{c}_5 u_5.$$

- If $a = 0$, then from Theorem 3.1, the general solution of system (3) is given by

  $$\tilde{Y}(t) = \begin{bmatrix} Q_p & Q_q \end{bmatrix} e^{J_{p+q}t} \tilde{Z}_{p+q}(0),$$

  where, $Q_p = \begin{bmatrix} u_1 & u_2 & u_3 \end{bmatrix}$, and $e^{\tilde{J}_{p+q}t}$ is given by (5). It is easy to observe that $\tilde{Y}_0 \in \text{colspan} \begin{bmatrix} Q_p & Q_q \end{bmatrix}$, and hence the solution is unique, and given by (6).

- If $c = 0$, , then from Theorem 3.1, the general solution of system (3) is given by

  $$\tilde{Y}(t) = Q_p e^{\tilde{J}_p t} \tilde{Z}_p(0).$$

  where, $Q_p = \begin{bmatrix} u_1 & u_2 & u_3 \end{bmatrix}$, and:

  $$e^{\tilde{J}_p t} = \begin{bmatrix} e^{\frac{-b+3d}{a}t} & 0 & 0 \\ 0 & e^{\frac{-b+2d}{a}t} & 0 \\ 0 & 0 & e^{\frac{-b+d}{a}t} \end{bmatrix}.$$

  It is easy to observe that $\tilde{Y}_0 \in \mathrm{colspanQ_p}$, and hence the solution is unique, and given by (6).

Numerical example 2

We consider now the system (1) with

$$E = \begin{bmatrix} 0 & -3 & 0 & 1 & 1 & 8 & 2 \\ 12 & 9 & -5 & -2 & -4 & -3 & 4 \\ 0 & -4 & -5 & 13 & 3 & 9 & 6 \\ 6 & -2 & -3 & 13 & 3 & -7 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & -11 & 26 & 4 & -1 & 8 \\ -3 & -3 & 2 & -1 & 1 & 3 & -1 \end{bmatrix}, A = \begin{bmatrix} 7 & 23 & 13 & 34 & 7 & -8 & 17 \\ -12 & -8 & 8 & 3 & 7 & 6 & -3 \\ -1 & 6 & -5 & -2 & 5 & -6 & -6 \\ 5 & 13 & 1 & -9 & -1 & 38 & 3 \\ 8 & 22 & 22 & 42 & 1 & 16 & 29 \\ 11 & 19 & 31 & 20 & 13 & 14 & 21 \\ 7 & 16 & 5 & 13 & 5 & -6 & 7 \end{bmatrix}.$$

By applying the Möbius transform (2) to the pencil of (1) we arrive at the family of systems (3). Let $\tilde{Y}_0 = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 1 & \end{bmatrix}^T$, be the initial conditions of (3). In this example we will focus on two cases:

(i) For $a = d = 0$, $b = d = 1$, we get the dual system of (1):

$$EY = AY';$$

(ii) For $a = c = d = 1$, $b = -1$, we get the discrete time system:

$$(E - A)Y_{k+1} = (E + A)Y_k, \quad k = 0, 1, 2, ...$$

We may now use Theorem 3.1, and the invariants of the pencil $sE - A$ of (1) in order to investigate the solutions of (i), (ii). The pencil $sE - A$ has three finite eigenvalues $s_1 = -1$, $s_2 = 0$, $s_3 = -2$ of algebraic multiplicity $p_1 = p_2 = p_3 = 1$, and an infinite eigenvalue of algebraic multiplicity $q = 4$. The eigenspaces of $sE - A$ associated with the eigenvalues -1, 0, -2, are :

$$\left\langle u_1 \right\rangle = \left\langle \begin{bmatrix} 0.5981 \\ -0.1773 \\ 0.2936 \\ 0.4628 \\ -0.2070 \\ 0.1516 \\ -1 \end{bmatrix} \right\rangle, \quad \left\langle u_2 \right\rangle = \left\langle \begin{bmatrix} -0.6466 \\ 0.1937 \\ -0.2696 \\ -0.4712 \\ 0.1885 \\ -0.1597 \\ 1 \end{bmatrix} \right\rangle, \quad \left\langle u_3 \right\rangle = \left\langle \begin{bmatrix} -0.6675 \\ 0.2118 \\ -0.3128 \\ -0.4433 \\ 0.1773 \\ -0.1872 \\ 1 \end{bmatrix} \right\rangle,$$

while the eigenspace of $sE - A$ associated with the infinite eigenvalue, including the generalized eigenvectors, is:

$$\left\langle u_4, u_5, u_6, u_7 \right\rangle = \left\langle \begin{bmatrix} 0.6283 \\ -0.1760 \\ 0.2854 \\ 0.4603 \\ -0.2130 \\ 0.1531 \\ -1 \end{bmatrix}, \begin{bmatrix} -0.6498 \\ 0.1802 \\ -0.3133 \\ -0.4687 \\ 0.1929 \\ -0.1480 \\ 1 \end{bmatrix}, \begin{bmatrix} -0.5969 \\ -0.1143 \\ 0.2738 \\ 0.4507 \\ -0.1623 \\ -0.1711 \\ -1 \end{bmatrix}, \begin{bmatrix} 0.6240 \\ -0.1652 \\ 0.2997 \\ 0.4692 \\ -0.2273 \\ 0.1578 \\ -1 \end{bmatrix} \right\rangle.$$

The Jordan matrix related to the finite eigenvalues, and the matrix H_q are:

$$J_p = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{bmatrix}, \quad H_q = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The matrix Q_q is defined as $Q_q = \begin{bmatrix} u_4 & u_5 & u_6 & u_7 \end{bmatrix}$. For the dual system (i), since $a = 0$ then from Theorem 3.1, the general solution of system (3) is given by

$$\tilde{Y}(t) = \begin{bmatrix} Q_p & Q_q \end{bmatrix} e^{J_{p+q}t} \tilde{Z}_{p+q}(0),$$

where $Q_p = \begin{bmatrix} u_1 & u_3 \end{bmatrix}$, and:

$$e^{\tilde{J}_{p+q}t} = \begin{bmatrix} e^{-t} & 0 & 0 & 0 & 0 & 0 \\ 0 & e^{-\frac{1}{2}t} & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

It is easy to observe that $\tilde{Y}_0 \notin \text{colspan}\begin{bmatrix} Q_p & Q_q \end{bmatrix}$, and hence there does not exist a unique solution. Since $\tilde{Z}_{p+q}(0)$ can not be defined uniquely, we set $\tilde{Z}_{p+q}(0) = \begin{bmatrix} c_1 & c_3 & c_4 & c_5 & c_6 & c_7 \end{bmatrix}$, and the general solution is:

$$\tilde{Y}(t) = e^{-t}c_1u_1 + e^{-\frac{1}{2}t}c_3u_3 + \sum_{i=4}^{7} c_iu_i.$$

For the discrete time system (ii), it is worth noting that since the steady state of (1) is neutral stable, the steady state of the discrete time system will be neutral stable as well. Furthermore, $a, c \neq 0$, and there does not exist an eigenvalue s_0 such that $a = s_0c$. Then from Theorem 3.1, it is easy to observe that $\tilde{Y}_0 \notin \text{colspan}\begin{bmatrix} Q_p & Q_q \end{bmatrix}$, and there does not exist a unique solution. Hence, since

$$\tilde{J}_{p+q} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{3} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix},$$

if we set $\tilde{Z}_{p+q}(0) = \begin{bmatrix} c_2 & c_3 & c_4 & c_5 & c_6 & c_7 \end{bmatrix}$, the general solution is given by:

$$\tilde{Y}_k = c_2u_2 + (-\frac{1}{3})^k c_3u_3 + (-1)^k \sum_{i=4}^{7} c_iu_i.$$

Numerical example 3

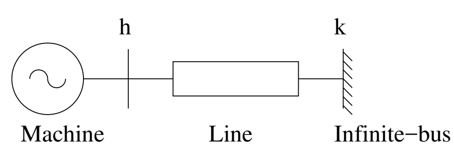

Figure 1: OMIB system.

Power systems can be modelled as a set of nonlinear differential algebraic equations, as follows:

$$\begin{aligned} T\dot{x} &= f(x, y) \\ R\dot{x} &= g(x, y) \ , \end{aligned}$$

where T, R are constant singular matrices, f are the differential equations, g are the algebraic equations, and $x \in \mathbb{R}^{n\times 1}$, $y \in \mathbb{R}^{m\times 1}$ are the state and algebraic variables, respectively. This semi-implicit formulation is a recently proposed alternative, see [24], to the explicit differential algebraic formulation $\dot{x} = f(x, y)$, $0 = g(x, y)$, which may be more familiar.

Eigenvalue analysis, which is a fundamental tool of the small-signal stability analysis of power systems, requires linearization around the operating point:

$$\begin{aligned} T\dot{x} &= f_x \Delta x + f_y \Delta y \\ R\dot{x} &= g_x \Delta x + g_y \Delta y \ , \end{aligned}$$

where f_x, f_y, g_x, g_y are the system Jacobian matrices. The linearized system can be equivalently represented as a singular system:

$$E_s \tilde{Y}' = A_s \tilde{Y}, \tag{7}$$

where $\tilde{Y} = \begin{bmatrix} x \\ y \end{bmatrix}$, $E_s = \begin{bmatrix} T & 0 \\ R & 0 \end{bmatrix}$, $A_s = \begin{bmatrix} f_x & f_y \\ g_x & g_y \end{bmatrix}$. The matrices E_s, A_s formulate a generalized eigenvalue problem. Special Möbius transforms (e.g., Cayley, shift-&-invert) are utilized by solvers to obtain the root loci of large scale power systems, especially by those that implement the Arnoldi iteration or any of its variants [22].

For illustration purposes, consider the simple example of a fourth order (two-axes) synchronous electrical machine connected through a transmission line to a slack bus, as shown in Fig. 1.

This system, which in power engineering is known as one-machine infinite-bus (OMIB) system, is commonly used for transient stability analysis studies. The term infinite is used as a consequence of the fact that the voltage at bus k is constant (infinite inertia). The OMIB system equations and variables are given in Table 1. The parameters of the system are given in Table 2.

The matrices that describe the linearized OMIB system are:

$$A_s = \begin{bmatrix}
\epsilon_1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \epsilon_2 & 0 & 0 & -0.41 & 0.23 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -0.70 & -0.73 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1.50 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1.15 & 0 & 0 \\
0 & 0 & 0 & 0 & \epsilon_3 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & \epsilon_4 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & \epsilon_5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & \epsilon_6 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 5.19 & -5.19 & 0.71 & 0.20 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -5.14 & 5.14 & -0.71 & -0.19 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0.20 & -0.20 & 4.89 & -5.04 & \epsilon_7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0.72 & -0.72 & -5.09 & 5.24 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \epsilon_8 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & \epsilon_9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & \epsilon_{10} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0.73 & 0 & 0 & 0 & 0 & 0 & -0.73 & 0 & 0.69 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
-0.70 & 0 & 0 & 0 & 0 & 0 & 0.70 & 0 & 0.72 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.30 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.55 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.23 & 0.41 & 0.70 & 0.73 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.41 & 0.23 & 0.73 & -0.70 & 0 & -1
\end{bmatrix},$$

$$E_s = \begin{bmatrix} 0.003 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 10 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

where ϵ_i represent components that $0 < |\epsilon_i| < 10^{-5}$ in order to simplify the notation.

Consider the special Möbius transform $a = -0.01$, $b = -0.01$, $c = -1$ and $d = 1$, which is a Cayley transformation. We apply our results to the transformed system (1), where $A = A_s + 0.01E_s$, $E = A_s - 0.01E_s$. This system has finite eigenvalues $s_1 = 0.8711$, $s_2 = 0.9997 + 0.0029i$, $s_3 = 0.9997 - 0.0029i$, $s_4 = 1$, $s_5 = 0,9963$ with respective algebraic multiplicities $p_1 = p_2 = p_3 = p_5 = 1$, $p_4 = 17$ and doesn't have any infinite eigenvalues ($q = 0$).

The original power system model is desrcibed by (7). In order to take information about the uniqueness of the solution of this system, we apply the inverse Möbius transform $z = \frac{-ds+b}{cs-a}$. In this case, $Q_p \in \mathbb{C}^{m\times p}$ are the linear independent eigenvectors (including the generalized) of all finite eigenvalues of $sE - A$ except the eigenvectors of $s_4 = 1$. Hence,

$$Q_p = \begin{bmatrix} -1 & 0.5822 + 0.0284i & 0.5822 - 0.0284i & -0.3163 \\ 4.62 \cdot 10^{-4} & -0.0005 - 0.0129i & -0.0005 + 0.0129i & 0.0055 \\ 0.8207 & 0.0031 - 0.0253i & 0.0031 + 0.0253i & -0.0194 \\ -0.4199 & 0.0017 - 0.0244i & 0.0017 + 0.0244i & -0.9614 \\ 0.7942 & 0.1038 + 0.1384i & 0.1038 - 0.1384i & 0.1459 \\ 0.6091 & -0.2689 - 0.0240i & -0.2689 + 0.0240i & 0.4185 \\ -0.0204 & -0.3240 - 0.0511i & -0.3240 + 0.0511i & 0.0827 \\ 0 & 0 & 0 & 0 \\ 0.1509 & 0.0305 + 0.0181i & 0.0305 - 0.0181i & -0.1846 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -0.7835 & -0.0355 - 0.1099i & -0.0355 + 0.1099i & 1 \\ -0.0028 & -0.8778 + 0.1135i & -0.8778 - 0.1135i & -0.2939 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -0.6091 & 0.3240 + 0.0511i & 0.3240 - 0.0511i & -0.4185 \\ 0.7942 & -0.2689 - 0.0240i & -0.2689 + 0.0240i & 0.1459 \\ 0.0882 & 0.9421 - 0.0012i & 0.9421 + 0.0012i & -0.5511 \\ -0.3440 & 0.4005 - 0.1586i & 0.4005 + 0.1586i & 0.9871 \\ 6.70 \cdot 10^{-4} & 0.8853 - 0.1147i & 0.8853 + 0.1147i & 0.2992 \\ 0.7346 & 0.1830 + 0.0836i & 0.1830 - 0.0836i & -0.8869 \end{bmatrix}.$$

Since $q = 0$, the matrices Q_q, H_q, J_q and $\tilde{J}_q$ do not exist. Therefore, there exists a unique solution of (7), if and only if:

$$\tilde{Y}_0 \in \mathrm{colspan}\ \mathrm{Q_p}.$$

In case that the condition for the existence of the solution does not hold, the system does not have an equilibrium point. The nonexistence of an equilibrium point in a power system is typically due to a saddle-node or limit-induced bifurcation point and indicates the collapse to the system's voltages due to its inability to deliver the power.

Conclusions

The aim of this article was to provide a link between the solutions of M-systems and an original system in the form (1), whose coefficients are square constant matrices and the leading coefficient singular. We proved that if the invariants of the pencil $sE - A$ are known, then it is possible to provide necessary and sufficient conditions for existence and uniqueness of solutions of the M-systems (3). This enables us to construct formulas for the solutions of each such system, when the solutions exist and are unique, without resorting to further processes of computations for each one separately. A further extension of this article is to study the connection of the solutions of singular systems of fractional nabla difference equations and their M-systems with a singular pencil and the connection of these kind of systems with singular systems of fractional differential equations. For all these there is already some research in progress, see [13], [14].

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Table 1: OMIB system equations and variables

Devices	Equations	Variables
Machine	$\frac{1}{\Omega_b}\dot{\delta} = \omega - \omega_s$	δ: rotor angle
	$2H\dot{\omega} = \tau_m - \tau_e - D(\omega - \omega_s)$	ω: angular speed
	$T'_{d0}\dot{e}'_q = -e'_q - (x_d - x'_d)i_d + v_f$	τ_m: mechanical torque
	$T'_{q0}\dot{e}'_d = -e'_d + (x_q - x'_q)i_q$	τ_e: electrical torque
	$0 = -p_h + v_d i_d + v_q i_q$	e'_q: q-axis transient emf
	$0 = -q_h + v_q i_d - v_d i_q$	i_d: d-axis current
	$0 = v_h \sin(\delta - \theta_h) - v_d$	v_f: field voltage
	$0 = v_h \cos(\delta - \theta_h) - v_q$	e'_d: d-axis transient emf
	$0 = -\tau_e + \psi_d i_q - \psi_q i_d$	i_q: q-axis current
	$0 = \tau_{m0} - \tau_m$	v_d: d-axis voltage
	$0 = v_{f0} - v_f$	v_q: q-axis voltage
	$0 = r_\alpha i_d + \psi_q + v_d$	v_h: voltage at bus h
	$0 = r_\alpha i_q - \psi_d + v_q$	θ_h: voltage angle at bus h
	$0 = v_q + r_\alpha i_q - e'_q + x'_d i_d$	ψ_q: q-axis magnetic flux
	$0 = v_d + r_\alpha i_d - e'_d - x'_q i_q$	ψ_d: d-axis magnetic flux
Line	$0 = -p_h + v_h^2(g_L + g_{L,h}) - v_h v_k(g_L \cos\theta_{hk} + b_L \sin\theta_{hk})$	p_h: active power injection at bus h
	$0 = -q_h - v_h^2(b_L + b_{L,h}) - v_h v_k(g_L \sin\theta_{hk} - b_L \cos\theta_{hk})$	q_h: reactive power injection at bus h
	$0 = -p_k + v_k^2(g_L + g_{L,h}) - v_h v_k(g_L \cos\theta_{hk} - b_L \sin\theta_{hk})$	p_k: active power injection at bus k
	$0 = -q_k - v_k^2(b_L + b_{L,h}) - v_h v_k(g_L \sin\theta_{hk} + b_L \cos\theta_{hk})$,	q_k: reactive power injection at bus k
	where $\theta_{hk} = \theta_h - \theta_k$.	
Infinite-bus	$0 = v_{G0,k} - v_k$	v_k: voltage at bus k
	$0 = \theta_{G0,k} - \theta_k$	θ_k: voltage angle at bus k

Table 2: OMIB system parameters

Device	Parameters
Machine	$\Omega_b = 314.16$ rad/s: base synchronous frequency, $\omega_s = 1$ pu [a] (rad/s): reference frequency, $H = 5$ MWs/MVA: inertia constant, $D = 0$ pu: damping coefficient, $T'_{d0} = 8$ s: d-axis transient time constant, $T'_{q0} = 0.4$ s: q-axis transient time constant, $x_d = 1.8$ pu (Ω): d-axis synchronous reactance, $x'_d = 0.3$ pu (Ω): d-axis transient reactance, $x_q = 1.7$ pu (Ω): q-axis synchronous reactance, $x'_q = 0.5$ pu (Ω): q-axis transient reactance, $\tau_{m0} = 0.46$ pu(MN·m): initial mechanical torque, $v_{f0} = 1.13$ pu (kV): initial field voltage, $r_\alpha = 0$ pu(Ω): armature resistance. $v_{G0,h} = 1.01$ pu (kV): initial voltage at bus h, $\theta_{G0,h} = 1.08°$: initial voltage angle at bus h.
Line	$r_L = 0.01$ pu (Ω): series resistance, $g_{L,h} = 0.04$ pu (Ω^{-1}): shunt conductance of sending-end h, $x_L = 0.2$ pu (Ω): series reactance, $b_{L,h} = 0$ pu (Ω^{-1}): shunt susceptance of sending-end h, where $g_L + jb_L = (r_L + jx_L)^{-1}$.
Infinite-bus	$v_{G0,k} = 1.03$ pu (kV): initial voltage at bus k, $\theta_{G0,k} = 0°$: initial voltage angle at bus k.

[a]per unit system (pu); in power system analysis, quantities are often expressed as fractions of defined base units.