Journal of Computational and Applied Mathematics 378 (2020) 112919

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

Generalized fractional controller for singular systems of differential equations

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ARTICLE INFO

Article history:
Received 23 November 2019
Received in revised form 29 March 2020

Keywords:
Singularity
Fractional calculus
Caputo derivative
Stability
Control

ABSTRACT

In this article we consider a class of singular linear systems of first order, and introduce a generalized fractional order feedback controller of Caputo type. The closed loop system in question is a singular system of differential equations having both first, and fractional order derivatives. We provide a comprehensive theory for the existence and uniqueness of solutions, as well as for the stability of the system with inclusion of the fractional order controller. An example of a singular system with a fractional order proportional integral controller, as well as an example on a 3-bus power system with inclusion of a fractional order damping controller, is given to illustrate our theory.

1. Introduction

Singular linear systems of differential & difference equations appear in control theory [1], circuit theory [2], and in modeling (dynamics) of electrical power systems [3]. Other interesting applications of a singular system are the constrained mechanical & robotic system described in [4], and in finance, the input–output Leontief model including its several important extensions [5].

In the last decade, many authors have studied problems of differential equations of fractional order, and have derived interesting results on different types of problems for given initial or boundary conditions, see [6–16]. Research has also been developed for other type of fractional operators such as the fractional nabla & delta operator applied to difference equations, see [12,17–24]. Focus has also been given on the mathematical modeling of many phenomena by using fractional operators. The theory of fractional differential equations (FDEs) is a promising tool for applications in physics [25], biology [26], and control theory, see [16,27–33]. Fractional-order operators are not just a generalization of the classical integer-order operators. Because of the way they are defined, more elaborated techniques are required for qualitative studies. In many practical cases the existing techniques are not enough.

Despite several studies, there are still parts missing for a complete and coherent theory of systems of FDEs in order to use this type of systems as a tool in the applied sciences in a similar way to the classical case. In addition, generalized FDEs and cases such as singularities of certain systems of FDEs have been mostly avoided in the framework of fractional calculus. Hence, explicit and easily testable methods are required in order to solve generalized systems of FDEs, so that applied researchers can redesign their models using fractional operators where this is appropriate.

The following notation is adopted throughout the paper. First order derivatives are indicated as $Y'(x) = \frac{d}{dx}Y(x)$; $\mathcal{L}$ denotes the Laplace transform [7]; and 0_{ij} indicates the zero matrix of i rows and j columns. Let $B_{n_1} \in \mathbb{C}^{n_1\times n_1}$, $B_{n_2} \in \mathbb{C}^{n_2\times n_2}$, $\dots, B_{n_r} \in \mathbb{C}^{n_r\times n_r}$. Then, the direct sum $B_{n_1} \oplus B_{n_2} \oplus \cdots \oplus B_{n_r}$ denotes the block diagonal matrix blockdiag $\left(B_{n_1}, B_{n_1}, \dots, B_{n_r}\right)$.

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https://doi.org/10.1016/j.cam.2020.112919

Definition 1.1 (*See [7]*)**.** Let $Y : [0, +\infty) \rightarrow \mathbb{R}^{m\times 1}$, $t \rightarrow Y$, denote a column of continuous and differentiable functions. Then, the Caputo (C) fractional derivative of order a, $0 < a < 1$, is defined by:

$$Y_C^{(a)}(t) := Y^{(a)}(t) = \frac{1}{\Gamma(1-a)} \int_0^t \left[(t-x)^{-a} Y'(x)\right] dx \ . \tag{1}$$

In this article, we consider the following singular system:

$$E_{pr} X'_{pr} = A_{pr} X_{pr} + B_{pr} U \ , \tag{2}$$

where $E_{pr}, A_{pr} \in \mathbb{C}^{\tilde{n}\times n}$ can be non-square, or square and singular, $X_{pr} \in \mathbb{C}^{n\times 1}$, $B_{pr} \in \mathbb{C}^{\tilde{n}\times\tilde{p}}$, $U \in \mathbb{C}^{\tilde{p}\times 1}$. Let the vector of the system output $\xi \in \mathbb{C}^{\tilde{q}\times 1}$, be:

$$\xi = C_{pr} X_{pr} + D_{pr} U \ , \tag{3}$$

where $C_{pr} \in \mathbb{C}^{\tilde{q}\times n}$, $D_{pr} \in \mathbb{C}^{\tilde{q}\times\tilde{p}}$. Then, a fractional order controller for the system (2)–(3), can be described by a set of fractional differential and algebraic equations as follows:

$$\begin{aligned} E_{c_1} {X_c}' + E_{c_2} {X_c}^{(\gamma)} &= A_c X_c + B_c \xi \ , \\ U &= C_c X_c + D_c \xi \ , \end{aligned} \tag{4}$$

where γ is the controller's fractional order derivative; $X_c \in \mathbb{R}^{\nu}$, is the vector of the controller states; E_c, $A_c \in \mathbb{C}^{\tilde{\nu}\times\nu}$, $B_c \in \mathbb{C}^{\tilde{\nu}\times\tilde{q}}$, $C_c \in \mathbb{C}^{\tilde{p}\times\nu}$, $D_c \in \mathbb{C}^{\tilde{p}\times\tilde{q}}$. Combining (2), (3), and (4) yields the closed-loop system representation. In matrix form:

$$\begin{bmatrix} E_{pr} & 0_{\tilde{n},\nu} & 0_{\tilde{n},\tilde{p}} \\ 0_{\tilde{\nu},n} & E_{c_1} & 0_{\tilde{\nu},\tilde{p}} \\ 0_{\tilde{p},n} & 0_{\tilde{p},\nu} & 0_{\tilde{p},\tilde{p}} \end{bmatrix} \begin{bmatrix} X_{pr} \\ X_c \\ U \end{bmatrix}' + \begin{bmatrix} 0_{\tilde{n},n} & 0_{\tilde{n},\nu} & 0_{\tilde{n},\tilde{p}} \\ 0_{\tilde{\nu},n} & E_{c_2} & 0_{\tilde{\nu},\tilde{p}} \\ 0_{\tilde{p},n} & 0_{\tilde{p},\nu} & 0_{\tilde{p},\tilde{p}} \end{bmatrix} \begin{bmatrix} X_{pr} \\ X_c \\ U \end{bmatrix}^{(\gamma)} =$$
$$\begin{bmatrix} A_{pr} & 0_{\tilde{n},\nu} & B_{pr} \\ B_c C_{pr} & A_c & B_c D_{pr} \\ D_c C_{pr} & C_c & D_c D_{pr} - I_{\tilde{p}} \end{bmatrix} \begin{bmatrix} X_{pr} \\ X_c \\ U \end{bmatrix} ,$$

or, equivalently,

$$E X' + \tilde{E} X^{(\gamma)} = A X \ , \tag{5}$$

where

$$E = \begin{bmatrix} E_{pr} & 0_{\tilde{n},\nu} & 0_{\tilde{n},\tilde{p}} \\ 0_{\tilde{\nu},n} & E_{c_1} & 0_{\tilde{\nu},\tilde{p}} \\ 0_{\tilde{p},n} & 0_{\tilde{p},\nu} & 0_{\tilde{p},\tilde{p}} \end{bmatrix}, \qquad X = \begin{bmatrix} X_{pr} \\ X_c \\ U \end{bmatrix},$$

and

$$\tilde{E} = \begin{bmatrix} 0_{\tilde{n},n} & 0_{\tilde{n},\nu} & 0_{\tilde{n},\tilde{p}} \\ 0_{\tilde{\nu},n} & E_{c_2} & 0_{\tilde{\nu},\tilde{p}} \\ 0_{\tilde{p},n} & 0_{\tilde{p},\nu} & 0_{\tilde{p},\tilde{p}} \end{bmatrix}, \qquad A = \begin{bmatrix} A_{pr} & 0_{\tilde{n},\nu} & B_{pr} \\ B_c C_{pr} & A_c & B_c D_{pr} \\ D_c C_{pr} & C_c & D_c D_{pr} - I_{\tilde{p}} \end{bmatrix}.$$

The article is organized as follows: in Section 2 we use the (C) fractional derivative as defined in (1), and study singular linear system of FDEs (5). We study the existence and uniqueness of solutions and provide two different types of formulas for the case that there exist solutions. In addition, we study stability properties, and finally, in Section 3 we provide numerical examples to justify our theory.

2. Main results

In this section, we present our main results. First, we provide the following property of the (C) fractional derivative [13]: Let $\phi(t)$, $\phi(t) \in \mathcal{C}^1[0, T]^{n\times 1}$ for some $T > 0$. Then, $[\phi^{(\alpha)}(t)]^{(\beta)} = [\phi^{(\beta)}(t)]^{(\alpha)} = \phi^{(\alpha+\beta)}(t)$, where $\alpha, \beta \in \mathbb{R}^+$, and $\alpha + \beta \leq 1$. We rewrite (5) as:

$$E X^{(\gamma+\beta)} + \tilde{E} X^{(\gamma)} = A X \ ,$$

where $\gamma + \beta = 1$. We use the notation:

$$\psi_1 = X \ , \qquad \psi_2 = X^{(\gamma)} \ .$$

Then, we obtain $\psi_1^{(\gamma)} = X^{(\gamma)} = \psi_2$, and $E\psi_2^{(\beta)} = EX' = -\tilde{E}\psi_2 + A\psi_1$. Or, equivalently:

$$\begin{bmatrix} I_{\tilde{\rho},\hat{\rho}} & 0_{\tilde{\rho},\hat{\rho}} \\ 0_{\tilde{\rho},\hat{\rho}} & E \end{bmatrix} \begin{bmatrix} \psi_1^{(\gamma)} \\ \psi_2^{(\beta)} \end{bmatrix} = \begin{bmatrix} 0_{\tilde{\rho},\hat{\rho}} & I_{\tilde{\rho},\hat{\rho}} \\ A & -\tilde{E} \end{bmatrix} \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} ,$$

where $\tilde{\rho} = \tilde{n} + \tilde{\nu} + \tilde{p}$, $\hat{\rho} = n + \nu + \tilde{p}$, and $I_{\tilde{\rho},\hat{\rho}}$ is a $\tilde{\rho} \times \hat{\rho}$ matrix with ones in its diagonal and zeros in the rest entries. Equivalently, we have:

$$FY^{\Delta}(t) = GY(t)\ , \tag{6}$$

where

$$F = \begin{bmatrix} I_{\tilde{\rho},\hat{\rho}} & 0_{\tilde{\rho},\hat{\rho}} \\ 0_{\tilde{\rho},\hat{\rho}} & E \end{bmatrix}, \qquad Y^{\Delta} = \begin{bmatrix} \psi_1^{(\gamma)} \\ \psi_2^{(\beta)} \end{bmatrix},$$

and

$$G = \begin{bmatrix} 0_{\tilde{\rho},\hat{\rho}} & I_{\tilde{\rho},\hat{\rho}} \\ A & -\tilde{E} \end{bmatrix}, \qquad Y = \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix},$$

where $F, G \in \mathbb{C}^{r\times m}$, $Y : [0, +\infty) \to \mathbb{C}^{m\times 1}$, and $\beta, \gamma \in (0, 1)$, where, for simplicity, $r = 2\tilde{\rho} = r$ and $m = 2\hat{\rho} = m$. The matrices F, G can be non-square ($r \neq m$) or square ($r = m$) with F singular (det(F)=0). Note that system (6) is equivalent to the closed loop system (5).

Definition 2.1. Assume $F, G \in \mathbb{C}^{r\times m}$, $\beta, \gamma \in (0, 1)$, an arbitrary $s \in \mathbb{C}$, and an inverse matrix function

$$z := z(s) = s^{\gamma} I_{\tilde{\rho}} \oplus s^{\beta} I_{\tilde{\rho}} = \begin{bmatrix} s^{\gamma} I_{\tilde{\rho}} & 0_{\tilde{\rho},\tilde{\rho}} \\ 0_{\tilde{\rho},\tilde{\rho}} & s^{\beta} I_{\tilde{\rho}} \end{bmatrix},$$

with $z \in \mathbb{C}^{r\times m}$. Then, the matrix pencil $zF - G$ is called:

1. *Regular* if $r = m$, i.e. F, G are square matrices, and $\det(zF - G) \not\equiv 0$;
2. *Singular* if
   - $r \neq m$, i.e. F, G are non-square matrices; or
   - $r = m$, i.e. F, G are square matrices, and $\det(zF - G) \equiv 0$.

Remark 2.1. Given $F, G \in \mathbb{C}^{r\times m}$, $\beta, \gamma \in (0, 1)$, an arbitrary $s \in \mathbb{C}$ and an inverse function $z = z(s) \in \mathbb{C}$, if the pencil $zF - G$ is:

(a) regular, since $\det(zF - G) \not\equiv 0$, there exists a matrix polynomial $\Theta(s) : \mathbb{C} \to \mathbb{R}^{m\times m}$ (which can be computed via the Gauss–Jordan elimination method, see [34]) such that:

$$\Theta(s)(zF - G) = \Lambda(s)\ , \tag{7}$$

where $\Lambda(s) : \mathbb{C} \to \mathbb{R}^{m\times m}$ is a diagonal matrix polynomial with non-zero elements;

(b) singular and $r > m$, there exists a matrix polynomial $\Theta(s) : \mathbb{C} \to \mathbb{R}^{r\times r}$ (which can be computed via the Gauss–Jordan elimination method) such that:

$$\Theta(s)(zF - G) = \begin{bmatrix} \Lambda(s) \\ 0_{r_1,m} \end{bmatrix}, \qquad \text{with} \qquad \Theta(s) = \begin{bmatrix} \Theta_1(s) \\ \Theta_2(s) \end{bmatrix}, \tag{8}$$

where $\Lambda(s) : \mathbb{C} \to \mathbb{R}^{m_1\times m}$, with $m_1 + r_1 = r$, is a matrix such that if $[l_{ij}]_{1\leq i\leq m_1}^{1\leq j\leq m}$ are its elements, for $i = j$ all elements are non-zero and for $i \neq j$ all elements are zero and $\Theta_1(s) \in \mathbb{R}^{m_1\times r}$, $\Theta_2(s) \in \mathbb{R}^{r_1\times r}$.

We now study the existence of solutions of system (6). We state the following Theorem:

Theorem 2.1. *Consider the system of FDEs (6), and let*

$$w := w(s) = s^{\gamma-1} I_{\tilde{\rho}} \oplus s^{\beta-1} I_{\tilde{\rho}} = \begin{bmatrix} s^{\gamma-1} I_{\tilde{\rho}} & 0_{\tilde{\rho},\tilde{\rho}} \\ 0_{\tilde{\rho},\tilde{\rho}} & s^{\beta-1} I_{\tilde{\rho}} \end{bmatrix}.$$

Then, there exist solutions for (6) if and only if either of the two following conditions is satisfied:

(a) the pencil of the system is regular; in this case, the general solution is given by:

$$Y(t) = \Phi(t)C\ , \tag{9}$$

where $\Phi(t) = \mathcal{L}^{-1}\{\Lambda^{-1}(s)\Theta(s)wF\}$, $\Lambda(s)$, $\Theta(s)$ are defined in (7) and $C \in \mathbb{R}^{m\times 1}$ is an unknown constant vector; or

(b) the pencil of the system is singular with $r > m$ and:

$$\Theta_2(s)F = 0_{m_1,1}, \qquad \text{and} \qquad m_1 = m. \tag{10}$$

In this case, the general solution is given by:

$$Y(t) = \Psi(t)C\ , \tag{11}$$

where $\Psi(t) = \mathcal{L}^{-1}\{\Lambda^{-1}(s)\Theta_1(s)wF\}$, $\Lambda(s)$, $\Theta(s)$, $\Theta_1(s)$, $\Theta_2(s)$ are defined in (8), $C \in \mathbb{R}^{m\times 1}$ is an unknown constant vector.

Proof. Let $\mathcal{L}\{Y(t)\} = Z(s)$, be the Laplace transform of $Y(t)$. Using the fractional derivative as defined in (1), by applying the Laplace transform $\mathcal{L}$ into (6), see [7,11], we get:

$$\mathcal{L}\{FY^{\Delta}(t)\} = \mathcal{L}\{GY(t)\} \ .$$

Note that

$$Y^{\Delta} = \begin{bmatrix} \frac{d^{\gamma}}{dt^{\gamma}} I_{\hat{\rho}} & 0_{\hat{\rho},\hat{\rho}} \\ 0_{\hat{\rho},\hat{\rho}} & \frac{d^{\beta}}{dt^{\beta}} I_{\hat{\rho}} \end{bmatrix} Y(t) \ ,$$

and hence

$$FY^{\Delta} = \begin{bmatrix} I_{\tilde{\rho},\hat{\rho}} & 0_{\tilde{\rho},\hat{\rho}} \\ 0_{\tilde{\rho},\hat{\rho}} & E \end{bmatrix} \begin{bmatrix} \frac{d^{\gamma}}{dt^{\gamma}} I_{\hat{\rho}} & 0_{\hat{\rho},\hat{\rho}} \\ 0_{\hat{\rho},\hat{\rho}} & \frac{d^{\beta}}{dt^{\beta}} I_{\hat{\rho}} \end{bmatrix} Y(t) \ ,$$

or, equivalently,

$$FY^{\Delta} = \begin{bmatrix} \frac{d^{\gamma}}{dt^{\gamma}} I_{\tilde{\rho},\hat{\rho}} & 0_{\tilde{\rho},\hat{\rho}} \\ 0_{\tilde{\rho},\hat{\rho}} & \frac{d^{\beta}}{dt^{\beta}} E \end{bmatrix} \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} = \begin{bmatrix} \frac{d^{\gamma}\psi_1}{dt^{\gamma}} & 0_{\tilde{\rho},\hat{\rho}} \\ 0_{\tilde{\rho},\hat{\rho}} & E\frac{d^{\beta}\psi_2}{dt^{\beta}} \end{bmatrix}.$$

Thus, $\mathcal{L}\{FY^{\Delta}(t)\} = F\mathcal{L}\{Y^{\Delta}(t)\}$. Furthermore:

$$F\mathcal{L}\left\{\begin{bmatrix} \psi_1^{(\gamma)}(t) \\ \psi_2^{(\beta)}(t) \end{bmatrix}\right\} = G\mathcal{L}\{Y(t)\} \ ,$$

or, equivalently,

$$F\begin{bmatrix} s^{\gamma}\mathcal{L}\{\psi_1(t)\} - s^{\gamma-1}\psi_1(0) \\ s^{\beta}\mathcal{L}\{\psi_2(t)\} - s^{\beta-1}\psi_2(0) \end{bmatrix} = G\mathcal{L}\{Y(t)\} \ ,$$

or, equivalently,

$$F\begin{bmatrix} s^{\gamma}\mathcal{L}\{\psi_1(t)\} \\ s^{\beta}\mathcal{L}\{\psi_2(t)\} \end{bmatrix} - F\begin{bmatrix} s^{\gamma-1}\psi_1(0) \\ s^{\beta-1}\psi_2(0) \end{bmatrix} = G\mathcal{L}\{Y(t)\} \ ,$$

or, equivalently,

$$F\begin{bmatrix} s^{\gamma}I_{\hat{\rho}} & 0_{\hat{\rho},\hat{\rho}} \\ 0_{\hat{\rho},\hat{\rho}} & s^{\beta}I_{\hat{\rho}} \end{bmatrix}\begin{bmatrix} \mathcal{L}\{\psi_1(t)\} \\ \mathcal{L}\{\psi_2(t)\} \end{bmatrix} - F\begin{bmatrix} s^{\gamma-1}I_{\hat{\rho}} & 0_{\hat{\rho},\hat{\rho}} \\ 0_{\hat{\rho},\hat{\rho}} & s^{\beta-1}I_{\hat{\rho}} \end{bmatrix}\begin{bmatrix} \psi_1(0) \\ \psi_2(0) \end{bmatrix} = G\mathcal{L}\{Y(t)\} \ ,$$

or, equivalently,

$$z(s)F\mathcal{L}\{Y(t)\} - w(s)FY(0) = G\mathcal{L}\{Y(t)\} \ ,$$

or, equivalently,

$$zFZ(s) - FwY(0) = GZ(s) \ ,$$

or, equivalently,

$$(zF - G)Z(s) = wFY_0 \ .$$

Where $Y_0 = Y(0)$, i.e. the initial condition of (6). Since we assume that Y_0 is unknown we can use an unknown constant vector $C \in \mathbb{C}^{m\times 1}$ and give to the above expression the following form:

$$(zF - G)Z(s) = wFC \ . \tag{12}$$

There are two cases. The first is (a) $r = m$ and det(zF − G) to be equal to a fractional polynomial with order less than $\beta + \gamma$ (regular pencil). The second case is (b) $r \neq m$ or $r = m$ and det(zF − G) ≡ 0, ∀ arbitrary $s \in \mathbb{C}$ (singular pencil).

In the case of (a), since the pencil is assumed regular and det(zF − G) $\not\equiv$ 0, there exists a matrix polynomial $\Theta(s) : \mathbb{C} \to \mathbb{R}^{m\times m}$ (which can be computed via the Gauss–Jordan elimination method, see [34]) such that:

$$\Theta(s)(zF - G) = \Lambda(s) \ ,$$

where $\Lambda(s) : \mathbb{C} \to \mathbb{R}^{m\times m}$ is a diagonal matrix polynomial with non-zero elements in its diagonal. Then, by multiplying (12) with $\Theta(s)$, we get:

$$\Theta(s)(zF - G)Z(s) = w\Theta(s)FC \ ,$$

or, equivalently,

$$\Lambda(s)Z(s) = w\Theta(s)FC \ ,$$

or, equivalently,

$$Z(s) = w\Lambda^{-1}(s)\Theta(s)FC \ .$$

The inverse Laplace transform of the matrix $w\Lambda^{-1}(s)\Theta(s)F = w(zF-G)^{-1}F$ always exists because its elements are fractions of fractional polynomials with the order of the polynomial in the denominator always being higher than the order of the polynomial in the numerator. Let $\mathcal{L}^{-1}\{w\Lambda^{-1}(s)\Theta(s)F\} = \Phi(t)$. Then, $Y(t)$ is given by (9).

In the case of (b), if $r < m$ there are at least $m - r$ unknown functions and m equations. Hence $Z(s)$ in system (12) cannot be defined uniquely. If $r > m$, there exists a matrix polynomial $\Theta(s) : \mathbb{C} \to \mathbb{R}^{r\times r}$ (which can be computed via the Gauss–Jordan elimination method), such that:

$$\Theta(s)(zF - G) = \begin{bmatrix} \Lambda(s) \\ 0_{r_1,m} \end{bmatrix},$$

where $\Lambda(s) : \mathbb{C} \to \mathbb{R}^{m_1\times m}$, with $m_1 + r_1 = r$, is a matrix such that if $[l_{ij}]_{1\le i\le m_1}^{1\le j\le m}$ are its elements, for $i = j$ all elements are non-zero and for $i \neq j$ all elements are zero. Let:

$$\Theta(s) = \begin{bmatrix} \Theta_1(s) \\ \Theta_2(s) \end{bmatrix},$$

where $\Theta_1(s) \in \mathbb{R}^{m_1\times r}$, $\Theta_2(s) \in \mathbb{R}^{r_1\times r}$. Then, system (12) has a unique solution if and only if (10) holds. In any other case, we have more unknown functions than equations or no solutions. If (10) holds, then:

$$\Theta(s)(zF - G) = \begin{bmatrix} \Lambda(s) \\ 0_{r_1,m} \end{bmatrix},$$

and we have

$$\Theta(s)(zF - G)Z(s) = \Theta(s)wFC \ ,$$

or, equivalently,

$$\Lambda(s)Z(s) = \Theta_1(s)wFC \ ,$$

or, equivalently,

$$Z(s) = \Lambda^{-1}(s)\Theta_1(s)wFC \ .$$

The inverse Laplace transform of $\Lambda^{-1}(s)\Theta_1(s)wF$ always exists because it is a matrix with elements fractions of fractional polynomials and with the order of the polynomial in the denominator always being higher than the order of the polynomial in the numerator. Let $\mathcal{L}^{-1}\{\Lambda^{-1}(s)\Theta_1(s)wF\} = \Psi(t)$. Then, $Y(t)$ is given (11). If $r = m$, there exists a matrix polynomial $\Theta(s) : \mathbb{C} \to \mathbb{R}^{r\times r}$ (which can be computed via the Gauss–Jordan elimination method) such that:

$$\Theta(s)(zF - G) = \Lambda(s) \oplus 0_{r_2,m_2} \ ,$$

where $\Lambda(s) : \mathbb{C} \to \mathbb{R}^{r_1\times m_1}$ with $r_1 \le m_1$ (because we apply Gauss–Jordan elimination method at the rows). All elements of $\Lambda(s)$ are zero except the ones in the diagonal with all non-zero elements. Also, $r_1 + r_2 = m_1 + m_2 = m$. Then, system (12) could have solutions if and only if $r_2 = m_2 = 0$, i.e. $r_1 = m_1 = m$; In any other case we have more unknown functions than equations or no solutions. But since we are in the case where $r = m$ and the pencil is singular, i.e. $\det(zF - G) \equiv 0$, this assumption can never hold. To sum up, there exist solutions for the system if the pencil is regular or singular with $r > m$ and $\Lambda(s)$ $m \times m$ and $\Theta_2(s)F = 0_{m-r,1}$. The proof is complete.

Having identified the conditions under which there exist solutions for singular systems in the form of (6), we can now present the following Remark:

Remark 2.2. For the (C) fractional derivative, if there exist solutions for system (6), then in the case that the pencil of the system is regular, the general solution is given by (9). In the case that the pencil of the system is singular with $r > m$ and (10) holds, the general solution is given by (11). In both cases, C is an unknown constant vector related to the initial conditions of the system since we used the Laplace transform. Note that:

(a) There are two types of initial conditions: *consistent*, which lead the system to have a unique solution; and *non-consistent*, which, if given, lead the system to have infinite solutions.
(b) It is not guaranteed that for given initial conditions a singular system of FDEs will have a unique solution. If the given initial conditions are consistent and there exist solutions for (6), we replace $C = Y_0$ in the formulas of the general solutions (9) and (11). However, if the given initial conditions are non-consistent but there exist solutions for (6), then the general solutions (9) and (11) hold for $t > 0$ and the system is impulsive.

(c) The next subsection provides a criterion on how to identify if the given initial conditions are consistent or non-consistent. For the case that the initial conditions are consistent, the matrix functions $\Phi(t)$ and $\Psi(t)$ can have elements defined for $t > 0$ and the columns $\Phi(t)Y_0$ and $\Psi(t)Y_0$ have all their elements always defined for $t \geq 0$.

Based on Theorem 2.1 and the assumptions for existence of solutions of system (6), we can provide additional formulas by using matrix pencil theory.

From Theorem 2.1, there exist solutions for system (6) if the pencil is either regular, or singular with $r > m$ and (10) holds. Hence, we focus on the case that $r \geq m$. If (6) has a *regular pencil*, then $sF - G$ is also a regular pencil. Hence and because of the structure of F there exist invariants of the following type:

- μ finite eigenvalues of algebraic multiplicity p_i, $i = 1, 2, \ldots, \mu$;
- an infinite eigenvalue of algebraic multiplicity q,

where $\sum_{i=1}^{\mu} p_i = p$, $p + q = m$. There exist non-singular matrices $P, Q \in \mathbb{C}^{m\times m}$ such that:

$$\begin{aligned} PFQ &= I_p \oplus H_q \ , \\ PGQ &= J_p \oplus I_q \ , \end{aligned} \tag{13}$$

where $J_p \in \mathbb{C}^{p\times p}$, $H_q \in \mathbb{C}^{q\times q}$ appropriate matrices with H_q a nilpotent matrix with index q_*, constructed by using the algebraic multiplicity of the infinite eigenvalue, J_p is a Jordan matrix, constructed by the finite eigenvalues of the pencil and their algebraic multiplicity. Let

$$P = \begin{bmatrix} P_{1,\gamma} \\ P_{1,\beta} \\ P_{2,\gamma} \\ P_{2,\beta} \end{bmatrix}, \qquad Q = \begin{bmatrix} Q_{p,\gamma} & Q_{p,\beta} & Q_{q,\gamma} & Q_{q,\beta} \end{bmatrix},$$

where $P_{1,\gamma} \in \mathbb{C}^{\hat{p}\times m}$, $P_{1,\beta} \in \mathbb{C}^{\bar{p}\times m}$, $P_{2,\gamma} \in \mathbb{C}^{\hat{q}\times m}$, $P_{2,\beta} \in \mathbb{C}^{\bar{q}\times m}$, and $Q_{p,\gamma} \in \mathbb{C}^{m\times\hat{p}}$, $Q_{p,\beta} \in \mathbb{C}^{m\times\bar{p}}$, $Q_{q,\gamma} \in \mathbb{C}^{m\times\hat{q}}$, $Q_{q,\beta} \in \mathbb{C}^{m\times\bar{q}}$. Equivalently, if we set:

$$P_1 = \begin{bmatrix} P_{1,\gamma} \\ P_{1,\beta} \end{bmatrix}, \qquad Q_p = \begin{bmatrix} Q_{p,\gamma} & Q_{p,\beta} \end{bmatrix},$$

$$P_2 = \begin{bmatrix} P_{2,\gamma} \\ P_{2,\beta} \end{bmatrix}, \qquad Q_q = \begin{bmatrix} Q_{q,\gamma} & Q_{q,\beta} \end{bmatrix},$$

we have

$$P = \begin{bmatrix} P_1 \\ P_2 \end{bmatrix}, \qquad Q = \begin{bmatrix} Q_p & Q_q \end{bmatrix}. \tag{14}$$

with $P_1 \in \mathbb{C}^{p\times m}$, $P_2 \in \mathbb{C}^{q\times m}$, and $Q_p \in \mathbb{C}^{m\times p}$, $Q_q \in \mathbb{C}^{m\times q}$. By using this notation, and $I_p = I_{\hat{p}} \oplus I_{\bar{p}}$, $J_p = J_{\hat{p}} \oplus J_{\bar{p}}$, (13) can be written in the following form:

$$\begin{aligned} PFQ &= I_{\hat{p}} \oplus I_{\bar{p}} \oplus H_{\hat{q}} \oplus H_{\bar{q}} \ , \\ PGQ &= J_{\hat{p}} \oplus J_{\bar{p}} \oplus I_{\hat{q}} \oplus I_{\bar{q}} \ . \end{aligned}$$

We provide the following theorem:

Theorem 2.2. *If there exist solutions for the system of FDEs (6), then:*

(a) Using the spectrum of the pencil $sF - G$, the general solution of (6) is given by:

$$Y(t) = Q_p \sum_{k=0}^{\infty} \begin{bmatrix} \frac{t^{\gamma k}}{\Gamma(k\gamma+1)} I_{\hat{p}} & 0_{\hat{p},\hat{p}} \\ 0_{\bar{p},\bar{p}} & \frac{t^{\beta k}}{\Gamma(k\beta+1)} I_{\bar{p}} \end{bmatrix} J_p^k C \ , \tag{15}$$

where $J_p \in \mathbb{C}^{p\times p}$, is a Jordan matrix constructed by the finite eigenvalues of the pencil $sF - G$, and their algebraic multiplicity, while $Q_p \in \mathbb{C}^{m\times p}$ is a matrix constructed by the linear independent eigenvectors related to the finite eigenvalues of the pencil $sF - G$, and $C \in \mathbb{C}^{p\times 1}$ is a constant vector.

(b) System (6) is asymptotically stable if all eigenvalues λ of the pencil $sF - G$ satisfy:

$$|\mathrm{Arg}(\lambda)| > \tilde{\gamma}\frac{\pi}{2} \ (\mathrm{rad}) \ , \tag{16}$$

where $\tilde{\gamma} = \min\{\gamma, 1-\gamma\}$.

Proof. For (a), we first observe that:

$$FY^{\Delta} = F \begin{bmatrix} \psi_1^{(\gamma)} \\ \psi_2^{(\beta)} \end{bmatrix} = F \begin{bmatrix} \frac{d^\gamma}{dt^\gamma} I_{\hat{\rho}} & 0_{\hat{\rho},\hat{\rho}} \\ 0_{\hat{\rho},\hat{\rho}} & \frac{d^\beta}{dt^\beta} I_{\hat{\rho}} \end{bmatrix} \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} = \begin{bmatrix} \frac{d^\gamma}{dt^\gamma} I_{\tilde{\rho}} & 0_{\tilde{\rho},\tilde{\rho}} \\ 0_{\tilde{\rho},\tilde{\rho}} & \frac{d^\beta}{dt^\beta} I_{\tilde{\rho}} \end{bmatrix} FY \ .$$

By using this notation we can write (6) in the form:

$$\begin{bmatrix} \frac{d^\gamma}{dt^\gamma} I_{\tilde{\rho}} & 0_{\tilde{\rho},\tilde{\rho}} \\ 0_{\tilde{\rho},\tilde{\rho}} & \frac{d^\beta}{dt^\beta} I_{\tilde{\rho}} \end{bmatrix} FY = GY \ ,$$

and by using the transformation $Y(t) = QZ(t)$ we obtain:

$$\begin{bmatrix} \frac{d^\gamma}{dt^\gamma} I_{\tilde{\rho}} & 0_{\tilde{\rho},\tilde{\rho}} \\ 0_{\tilde{\rho},\tilde{\rho}} & \frac{d^\beta}{dt^\beta} I_{\tilde{\rho}} \end{bmatrix} FQZ = GQZ \ ,$$

whereby, multiplying by P and using (13) and (14), we get:

$$(I_{\hat{p}} \oplus I_{\bar{p}} \oplus H_{\hat{q}} \oplus H_{\bar{q}}) \begin{bmatrix} Z_{\hat{p}}^{(\gamma)}(t) \\ Z_{\bar{p}}^{(\beta)}(t) \\ Z_{\hat{q}}^{(\gamma)}(t) \\ Z_{\bar{q}}^{(\beta)}(t) \end{bmatrix} = (J_{\hat{p}} \oplus J_{\bar{p}} \oplus I_{\hat{q}} \oplus I_{\bar{q}}) \begin{bmatrix} Z_{\hat{p}}(t) \\ Z_{\bar{p}}(t) \\ Z_{\hat{q}}(t) \\ Z_{\bar{q}}(t) \end{bmatrix},$$

where

$$Z(t) = \begin{bmatrix} Z_{\hat{p}}(t) \\ Z_{\bar{p}}(t) \\ Z_{\hat{q}}(t) \\ Z_{\bar{q}}(t) \end{bmatrix},$$

with $Z_{\hat{p}}(t) \in \mathbb{C}^{\hat{p}\times 1}$, $Z_{\bar{p}}(t) \in \mathbb{C}^{\bar{p}\times 1}$, $Z_{\hat{q}}(t) \in \mathbb{C}^{\hat{q}\times 1}$, $Z_{\bar{q}}(t) \in \mathbb{C}^{\bar{q}\times 1}$. From the above expressions we arrive at the subsystems:

$$\begin{aligned} Z_{\hat{p}}^{(\gamma)}(t) &= J_{\hat{p}} Z_{\hat{p}}(t) \ ; \\ Z_{\bar{p}}^{(\beta)}(t) &= J_{\bar{p}} Z_{\bar{p}}(t) \ , \end{aligned} \tag{17}$$

and

$$\begin{aligned} H_{\hat{q}} Z_{\hat{q}}^{(\gamma)}(t) &= Z_{\hat{q}}(t) \ ; \\ H_{\bar{q}} Z_{\bar{q}}^{(\beta)}(t) &= Z_{\bar{q}}(t) \ . \end{aligned} \tag{18}$$

By applying the Laplace transform $\mathcal{L}$ into (17), we get:

$$\begin{aligned} \mathcal{L}\{Z_{\hat{p}}^{(\gamma)}(t)\} &= J_{\hat{p}} \mathcal{L}\{Z_{\hat{p}}(t)\} \ ; \\ \mathcal{L}\{Z_{\bar{p}}^{(\beta)}(t)\} &= J_{\bar{p}} \mathcal{L}\{Z_{\bar{p}}(t)\} \ , \end{aligned}$$

Let $\mathcal{L}\{Z_{\hat{p}}(t)\} = W_{\hat{p}}(s)$, $\mathcal{L}\{Z_{\bar{p}}(t)\} = W_{\bar{p}}(s)$, and $Z_{\hat{p}0} = Z_{\hat{p}}(0)$, $Z_{\bar{p}0} = Z_{\bar{p}}(0)$, i.e. the initial condition of (17). Since we assume that $Z_{\hat{p}}(0)$, $Z_{\bar{p}}(0)$ are unknown, we set $Z_{\hat{p}}(0) = C_1$, $Z_{\bar{p}}(0) = C_2$, where C_1, C_2 unknown columns, and give to the above expression the following form:

$$\begin{aligned} (s^\gamma I_{\hat{p}} - J_{\hat{p}}) W_{\hat{p}}(s) &= s^{\gamma-1} C_1 \ ; \\ (s^\beta I_{\bar{p}} - J_{\bar{p}}) W_{\bar{p}}(s) &= s^{\beta-1} C_2 \ , \end{aligned}$$

or, equivalently,

$$\begin{aligned} W_{\hat{p}}(s) &= s^{\gamma-1} (s^\gamma I_{\hat{p}} - J_{\hat{p}})^{-1} C_1 \ ; \\ W_{\bar{p}}(s) &= s^{\beta-1} (s^\beta I_{\bar{p}} - J_{\bar{p}})^{-1} C_2 \ . \end{aligned}$$

By taking into account that $(s^\gamma I_{\hat{p}} - J_{\hat{p}})^{-1} = \sum_{k=0}^{\infty} s^{-(k+1)\gamma} J_{\hat{p}}^k$, and $(s^\beta I_{\bar{p}} - J_{\bar{p}})^{-1} = \sum_{k=0}^{\infty} s^{-(k+1)\beta} J_{\bar{p}}^k$, we have:

$$\begin{aligned} W_{\hat{p}}(s) &= \textstyle\sum_{k=0}^{\infty} s^{-\gamma k-1} J_{\hat{p}}^k C_1 \ ; \\ W_{\bar{p}}(s) &= \textstyle\sum_{k=0}^{\infty} s^{-\beta k-1} J_{\bar{p}}^k C_2 \ . \end{aligned} \tag{19}$$

Then:

$$\begin{aligned} Z_{\hat{p}}(t) &= \textstyle\sum_{k=0}^{\infty} \frac{t^{\gamma k}}{\Gamma(k\gamma+1)} J_{\hat{p}}^k C_1 \ ; \\ Z_{\bar{p}}(t) &= \textstyle\sum_{k=0}^{\infty} \frac{t^{\beta k}}{\Gamma(k\beta+1)} J_{\bar{p}}^k C_2 \ . \end{aligned} \tag{20}$$

Let q_* be the index of the nilpotent matrix $H_{\bar{q}}$, i.e. $H_{\bar{q}}^{q_*} = 0_{\bar{q},\bar{q}}$. Then, if we obtain the following matrix equations:

$$\begin{aligned} H_{\bar{q}} Z_q^{(\beta)}(t) &= Z_q(t) \\ H_{\bar{q}}^2 Z_q^{(2\beta)}(t) &= H_{\bar{q}} Z_q^{(\beta)}(t) \\ H_{\bar{q}}^3 Z_q^{(3\beta)}(t) &= H_{\bar{q}}^2 Z_q^{(2\beta)}(t) \\ H_{\bar{q}}^4 Z_q^{(4\beta)}(t) &= H_{\bar{q}}^3 Z_q^{(3\beta)}(t) \\ &\vdots \\ H_{\bar{q}}^{q_*-1} Z_q^{([q_*-1]\beta)}(t) &= H_{\bar{q}}^{q_*-2} Z_q^{([q_*-2]\beta)}(t) \\ H_{\bar{q}}^{q_*} Z_q^{(q_*\beta)}(t) &= H_{\bar{q}}^{q_*-1} Z_q^{([q_*-1]\beta)}(t) \end{aligned} ,$$

by taking the sum of the above equations and using the fact that $H_{\bar{q}}^{q_*} = 0_{\bar{q},\bar{q}}$, we arrive at $H_{\bar{q}} = 0_{\bar{q},1}$. Similarly, $H_{\hat{q}} = 0_{\hat{q},1}$. Hence, the solution of the subsystem (18) is:

$$Z_q(t) = 0_{q,1} \ . \tag{21}$$

To conclude, by combining (20) and (21), for the case of a regular pencil, system (6) has the solution:

$$Y(t) = QZ(t) = \begin{bmatrix} Q_{p,\gamma} & Q_{p,\beta} & Q_q \end{bmatrix} \begin{bmatrix} \sum_{k=0}^{\infty} \frac{t^{\gamma k}}{\Gamma(k\gamma+1)} J_{\hat{p}}^k C_1 \\ \sum_{k=0}^{\infty} \frac{t^{\beta k}}{\Gamma(k\beta+1)} J_{\bar{p}}^k C_2 \\ 0_{q,1} \end{bmatrix},$$

or, equivalently,

$$Y(t) = Q_{p,\gamma} \sum_{k=0}^{\infty} \frac{t^{\gamma k}}{\Gamma(k\gamma+1)} J_{\hat{p}}^k C_1 + Q_{p,\beta} \sum_{k=0}^{\infty} \frac{t^{\beta k}}{\Gamma(k\beta+1)} J_{\bar{p}}^k C_2 \ .$$

From here, we arrive at (15). Next, we consider system (6) with a *singular pencil* and $r > m$. In general, the class of $sF - G$ is then characterized by a uniquely defined element, known as the complex Kronecker canonical form, see [35], specified by the complete set of invariants of the singular pencil $sF - G$. This is the set of the finite–infinite eigenvalues and the minimal column–row indices. In the case of $r > m$, there exist only row minimal indices. Let $\mathcal{N}_l$ be the left null space of a matrix. Then, the equations $V^T(s)(sF - G) = 0_{1,m}$, have solutions in $V(s)$, which are vectors in the rational vector spaces $\mathcal{N}_l(sF - G)$. The binary vectors $V^T(s)$ express dependence relationships among the rows of $sF - G$. Note that $V(s) \in \mathbb{C}^{r\times 1}$ are polynomial vectors. Let $t = \dim\mathcal{N}_l(sF - G)$. It is known that $\mathcal{N}_l(sF - G)$, as rational vector spaces, are spanned by minimal polynomial bases of minimal degrees:

$$\varsigma_1 = \varsigma_2 = \cdots = \varsigma_h = 0 < \varsigma_{h+1} \le \cdots \le \varsigma_{h+k} \ ,$$

which is the set of *row minimal indices* of $sF - G$. This means there are $h + k$ row minimal indices, but k non-zero row minimal indices. We are interested only in the k non zero minimal indices. To sum up, the invariants of a singular pencil with $r > m$ are the finite–infinite eigenvalues of the pencil and the minimal row indices as described above. Following the above given analysis, there exist non-singular matrices P, Q with $P \in \mathbb{C}^{r\times r}$, $Q \in \mathbb{C}^{m\times m}$, such that:

$$\begin{aligned} PFQ &= F_K = I_p \oplus H_q \oplus F_\varsigma \ , \\ PGQ &= G_K = J_p \oplus I_q \oplus G_\varsigma \ , \end{aligned} \tag{22}$$

where J_p is the Jordan matrix for the finite eigenvalues, H_q a nilpotent matrix with index q_* which is actually the Jordan matrix of the zero eigenvalue of the pencil $sG - F$. The matrices F_ς, G_ς are defined as:

$$\begin{aligned} F_\varsigma &= \begin{bmatrix} I_{\varsigma_{h+1}} \\ 0_{1,\varsigma_{h+1}} \end{bmatrix} \oplus \begin{bmatrix} I_{\varsigma_{h+2}} \\ 0_{1,\varsigma_{h+2}} \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} I_{\varsigma_{h+k}} \\ 0_{1,\varsigma_{h+k}} \end{bmatrix}, \\ G_\varsigma &= \begin{bmatrix} 0_{1,\varsigma_{h+1}} \\ I_{\varsigma_{h+1}} \end{bmatrix} \oplus \begin{bmatrix} 0_{1,\varsigma_{h+2}} \\ I_{\varsigma_{h+2}} \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} 0_{1,\varsigma_{h+k}} \\ I_{\varsigma_{h+k}} \end{bmatrix}, \end{aligned} \tag{23}$$

with $p + q + \sum_{i=1}^{k}[\varsigma_{h+i}] + k = r$, $p + q + \sum_{i=1}^{k}[\varsigma_{h+i}] = m$. Finally, the matrices P, Q can be written as:

$$P = \begin{bmatrix} P_1 \\ P_2 \\ P_3 \end{bmatrix}, \qquad Q = \begin{bmatrix} Q_p & Q_q & Q_\varsigma \end{bmatrix}, \tag{24}$$

with $P_1 \in \mathbb{C}^{p\times r}$, $P_2 \in \mathbb{C}^{q\times r}$, $P_3 \in \mathbb{C}^{\varsigma_1\times r}$, $\varsigma_1 = k + \sum_{i=1}^{k}[\varsigma_{h+i}]$ and $Q_p \in \mathbb{C}^{m\times p}$, $Q_q \in \mathbb{C}^{m\times q}$, $Q_\varsigma \in \mathbb{C}^{m\times\varsigma_2}$ and $\varsigma_2 = \sum_{i=1}^{k}[\varsigma_{h+i}]$.

By substituting the transformation $Y(t) = QZ(t)$ into (6), multiplying by P, using (22), (24) and setting $Z(t) = \begin{bmatrix} Z_p(t) \\ Z_q(t) \\ Z_\varsigma(t) \end{bmatrix}$, $Z_p(t) \in \mathbb{C}^{p\times 1}$, $Z_p(t) \in \mathbb{C}^{q\times 1}$ and $Z_\varsigma(t) \in \mathbb{C}^{\varsigma_2\times 1}$, we arrive at the subsystems (17), (18), and:

$$F_\varsigma Z_\varsigma^{(\beta)}(t) = G_\varsigma Z_\varsigma(t)\,. \tag{25}$$

For the subsystem (25), let

$$Z_\varsigma(t) = \begin{bmatrix} Z_{\varsigma_{h+1}}(t) \\ Z_{\varsigma_{h+2}}(t) \\ \vdots \\ Z_{\varsigma_{h+k}}(t) \end{bmatrix}, \quad Z_{\varsigma_{h+i}}(t) \in \mathbb{C}^{(\varsigma_{h+i})\times 1}, \quad i = 1, 2, \ldots, k \tag{26}$$

with

$$Z_{\varsigma_{h+i}}(t) = \begin{bmatrix} Z_{\varsigma_{h+i},1}(t) \\ Z_{\varsigma_{h+i},2}(t) \\ \vdots \\ Z_{\varsigma_{h+i},\varsigma_{h+i}}(t) \end{bmatrix}. \tag{27}$$

By replacing (23) into (25), we get:

$$\begin{bmatrix} I_{\varsigma_{h+i}} \\ 0_{1,\varsigma_{h+i}} \end{bmatrix} Z^{(\beta)}_{\varsigma_{h+i}}(t) = \begin{bmatrix} 0_{1,\varsigma_{h+i}} \\ I_{\varsigma_{h+i}} \end{bmatrix} Z_{\varsigma_{h+i}}(t)\,,$$

or, equivalently, by using the above expressions:

$$\begin{bmatrix} 1 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \ldots & \vdots \\ 0 & 0 & \ldots & 1 \\ 0 & 0 & \ldots & 0 \end{bmatrix} \begin{bmatrix} Z^{(\beta)}_{\varsigma_{h+i},1}(t) \\ Z^{(\beta)}_{\varsigma_{h+i},2}(t) \\ \vdots \\ Z^{(\beta)}_{\varsigma_{h+i},\varsigma_{h+i}}(t) \end{bmatrix} = \begin{bmatrix} 0 & 0 & \ldots & 0 \\ 1 & 0 & \ldots & 0 \\ \vdots & \vdots & \ldots & \vdots \\ 0 & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 1 \end{bmatrix} \begin{bmatrix} Z_{\varsigma_{h+i},1}(t) \\ Z_{\varsigma_{h+i},2}(t) \\ \vdots \\ Z_{\varsigma_{h+i},\varsigma_{h+i}}(t) \end{bmatrix},$$

or, equivalently,

$$\begin{aligned} Z^{(\beta)}_{\varsigma_{h+i},1}(t) &= 0 \\ Z^{(\beta)}_{\varsigma_{h+i},2}(t) &= Z_{\varsigma_{h+i},1}(t) \\ &\vdots \\ Z^{(\beta)}_{\varsigma_{h+i},\varsigma_{h+i}}(t) &= Z_{\varsigma_{h+i},\varsigma_{h+i}-1}(t) \\ 0 &= Z_{\varsigma_{h+i},\varsigma_{h+i}}(t) \end{aligned}.$$

We have a system of ς_{h+i}+1 FDEs and ς_{h+i} unknowns. Starting from the last equation, we get the solutions:

$$\begin{aligned} Z_{\varsigma_{h+i},\varsigma_{h+i}}(t) &= 0 \\ Z_{\varsigma_{h+i},\varsigma_{h+i}-1}(t) &= 0 \\ Z_{\varsigma_{h+i},\varsigma_{h+i}-2}(t) &= 0 \\ &\vdots \\ Z_{\varsigma_{h+i},1}(t) &= 0\,. \end{aligned} \tag{28}$$

In order to solve the system we used the last ς_j equations and concluded that system (25) has the zero solution. Hence, also in this case, the solutions are given by (15). For (b) we can rewrite the Jordan matrix in the form:

$$J_{\hat{p}} := J_{\hat{p}_1}(\lambda_1) \oplus \ldots \oplus J_{\hat{p}_\mu}(\lambda_\mu)\,,$$

$$J_{\tilde{p}} := J_{\tilde{p}_1}(\lambda_1) \oplus \cdots \oplus J_{\tilde{p}_\mu}(\lambda_\mu)\,,$$

where

$$J_{p_i}(\lambda_i) = \begin{bmatrix} \lambda_i & 1 & \ldots & 0 & 0 \\ 0 & \lambda_i & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & \lambda_i & 1 \\ 0 & 0 & \ldots & 0 & \lambda_i \end{bmatrix} \in \mathbb{C}^{p_i\times p_i}, \quad i = 1, 2, \ldots, \mu.$$

In addition, the matrix Q_p can be written in the form:

$$Q_{\hat{p}} = \begin{bmatrix} u_{1,\hat{p}_1} & \dots & u_{1,2} & u_{1,1} & \dots & u_{\mu,\hat{p}_\mu} & \dots & u_{\mu,2} & u_{\mu,1} \end{bmatrix} ,$$

$$Q_{\bar{p}} = \begin{bmatrix} v_{1,\bar{p}_1} & \dots & v_{1,2} & v_{1,1} & \dots & v_{\mu,\bar{p}_\mu} & \dots & v_{\mu,2} & v_{\mu,1} \end{bmatrix} ,$$

where $u_{i,j}$, $j = 1, 2, \dots, \hat{p}_i$ linear independent eigenvectors, $v_{i,j}$, $j = 1, 2, \dots, \bar{p}_i$ linear independent eigenvectors of λ_i, $i = 1, 2, \dots, \mu$. Furthermore, C_1, C_2 can be written as:

$$C_1 = \begin{bmatrix} c_{1,\hat{p}_1} & \dots & c_{1,2} & c_{1,1} & \dots & c_{\mu,\hat{p}_\mu} & \dots & c_{\mu,2} & c_{\mu,1} \end{bmatrix}^T ,$$

$$C_2 = \begin{bmatrix} c_{1,\bar{p}_1} & \dots & c_{1,2} & c_{1,1} & \dots & c_{\mu,\bar{p}_\mu} & \dots & c_{\mu,2} & c_{\mu,1} \end{bmatrix}^T ,$$

where $c_{i,j} \in \mathbb{C}$, $i = 1, 2, \dots, \mu$, $j = 1, 2, \dots, \hat{p}_i$, $d_{i,j} \in \mathbb{C}$, $i = 1, 2, \dots, \mu$, $j = 1, 2, \dots, \bar{p}_i$constants. If we replace the above expressions in the general solution we arrive at:

$$Y(t) = \sum_{i=1}^{\mu} \sum_{k=0}^{\infty} \frac{(\sqrt[\gamma]{\lambda_i} t)^{\gamma k}}{\Gamma(k\gamma + 1)} \sum_{j=1}^{\hat{p}_i} \Big(\sum_{k=1}^{j} c_{i,j-(k-1)} t^{k-1} \Big) u_{i,j} +$$
$$\sum_{i=1}^{\mu} \sum_{k=0}^{\infty} \frac{(\sqrt[\beta]{\lambda_i} t)^{\beta k}}{\Gamma(k\beta + 1)} \sum_{j=1}^{\bar{p}_i} \Big(\sum_{k=1}^{j} d_{i,j-(k-1)} t^{k-1} \Big) v_{i,j} ,$$

where $c_{i,j-(k-1)}, d_{i,j-(k-1)} \in \mathbb{C}$, constants. From the transformation $Y(t) = QZ(t)$ from the proof in (a), we have $Y(t) = Q_p Z_p(t)$ or, equivalently,

$$Y = Q_p Z_p .$$

From (14) we have that $P_1 E Q_p = I_p$. By multiplying the above expression by $P_1 E$ we have:

$$P_1 E Y = P_1 E Q_p Z_p ,$$

or, equivalently,

$$Z_p = P_1 E Y .$$

Hence:

$$Z_p(0) = P_1 E Y(0) .$$

From (14), we have:

$$P_1 = \begin{bmatrix} P_{1,\gamma} \\ P_{1,\beta} \end{bmatrix}.$$

We set

$$P_1 = \begin{bmatrix} w_{1,\hat{p}_1} \\ \vdots \\ w_{1,2} \\ w_{1,1} \\ \vdots \\ w_{\mu,\hat{p}_\mu} \\ \vdots \\ w_{\mu,2} \\ w_{\mu,1} \end{bmatrix}, \qquad P_2 = \begin{bmatrix} \omega_{1,\bar{p}_1} \\ \vdots \\ \omega_{1,2} \\ \omega_{1,1} \\ \vdots \\ \omega_{\mu,\bar{p}_\mu} \\ \vdots \\ \omega_{\mu,2} \\ \omega_{\mu,1} \end{bmatrix},$$

where $w_{i,j}$, $j = 1, 2, \dots, \hat{p}_i$, $\omega_{i,j}$, $j = 1, 2, \dots, \bar{p}_i$ linear independent left eigenvectors of λ_i, $i = 1, 2, \dots, \mu$. By replacing the above expressions into the general solution we arrive at:

$$Y(t) = \sum_{i=1}^{\mu} \sum_{k=0}^{\infty} \frac{(\sqrt[\gamma]{\lambda_i} t)^{\gamma k}}{\Gamma(k\gamma + 1)} \sum_{j=1}^{\hat{p}_i} \Big(\sum_{k=1}^{j} t^{k-1} w_{i,j-(k-1)} E Y(0) \Big) u_{i,j} +$$
$$\sum_{i=1}^{\mu} \sum_{k=0}^{\infty} \frac{(\sqrt[\beta]{\lambda_i} t)^{\beta k}}{\Gamma(k\beta + 1)} \sum_{j=1}^{\bar{p}_i} \Big(\sum_{k=1}^{j} t^{k-1} \omega_{i,j-(k-1)} E Y(0) \Big) v_{i,j}.$$

System (6) is asymptotically stable if $\lim_{t\to+\infty} Y(t) = 0_{m,1}$. This holds if all eigenvalues λ of the pencil $sF - G$ satisfy:

$$\max\left\{\frac{\mathrm{Arg}(\lambda)}{\gamma}, \frac{\mathrm{Arg}(\lambda)}{1-\gamma}\right\} > \frac{\pi}{2}\ (\mathrm{rad})\ ,$$

or, equivalently (16) holds. The proof is complete.

Having identified the conditions under which there exist solutions for singular systems in the form of (6), we can now present the following Corollary:

Corollary 2.1. *If there exist solutions for system (6), then in the case that:*

(a) The pencil of the system is regular;
(b) The pencil of the system is singular with $r > m$.

Then, for given initial conditions $Y(t_0) = Y_0$, the solution is unique if and only if:

$$Y_0 \in \mathrm{colspan}(Q_p)\ . \tag{29}$$

The solution is then given by (15) and C is the unique solution of the linear system:

$$Q_p C = Y_0\ . \tag{30}$$

Proof. This is a direct result from Theorem 2.2. For both (a), (b) if we use the formula (15) for $t = 0$ we get:

$$Y(0) = Q_p C\ ,$$

and we arrive at condition (29) because C is assumed an unknown vector. The above linear system has always a unique solution for C since the matrix Q_p has linear independent columns. The proof is complete.

Theorem 2.2 can be used to further develop methods to measure the participation of system eigenvalues in system states [36] for singular linear systems of FDEs as well as for the extension of linear fractional operators, see [37], and their applications into electrical power systems, see [38].

3. Numerical examples

In this section, we provide two examples. In the first example, we exploit the main results of this article by studying a small linear singular system of differential equations with regular pencil. In the second example, we study the damping of the electro-mechanical oscillations of a 3-bus power system through a fractional order power system stabilizer.

3.1. Numerical example 1

We consider system (2) with:

$$E_{pr} = \begin{bmatrix} 4 & 9 & 9 & -2 & 10 & 7 & 3 \\ 1 & 5 & 2 & 2 & 3 & 1 & 1 \\ 1 & 0 & -2 & -2 & 6 & 4 & 1 \\ 5 & -2 & -3 & 18 & 3 & 16 & 2 \\ 6 & 8 & 6 & 8 & 6 & 14 & 2 \\ 2 & 11 & 3 & 6 & 6 & 2 & 2 \\ 4 & 5 & 5 & 6 & 2 & 9 & 1 \end{bmatrix},$$

$$A_{pr} = \begin{bmatrix} -15 & -43 & -39 & 4 & -35 & -22 & -5 \\ -3 & -19 & -2 & -5 & -12 & -2 & 1 \\ -4 & -16 & -30 & 6 & -9 & -1 & -7 \\ -25 & -2 & 3 & -72 & -3 & -74 & -4 \\ -27 & -32 & -23 & -39 & -24 & -66 & -5 \\ -8 & -41 & -15 & -18 & -24 & -8 & -8 \\ -18 & -15 & -9 & -29 & -13 & -48 & -1 \end{bmatrix}, \qquad B_{pr} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}.$$

The matrix pencil $sE_{pr} - sA_{pr}$ has $p = 5$ finite, distinct eigenvalues $\lambda_1 = -5$, $\lambda_2 = -4$, $\lambda_3 = 1$, $\lambda_4 = -2$ and $\lambda_5 = -3$. The pencil also has the eigenvalue $\lambda_6 = \infty$ with algebraic multiplicity $q = 2$. The rightmost eigenvalue of the pencil is $\lambda_3 > 0$, and thus the system is unstable. The output of the system is given by (3), where:

$$C_{pr} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \qquad D_{pr} = 0\ .$$

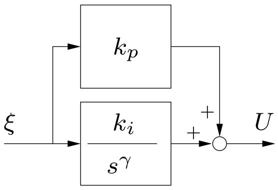

Fig. 1. FOPI block diagram.

In order to stabilize the system, we consider the following simple form of controller (4):

$$\begin{aligned} X_c{}^{(\gamma)} &= k_i\xi \ , \\ U &= X_c + k_p\xi \ . \end{aligned} \tag{31}$$

Eq. (31) describes a fractional order proportional–integral controller (FOPI), where $k_p = 7$, $k_i = 10$, are the proportional and integral gains, respectively; $\gamma = 0.6$ is the controller's fractional order. The block diagram of the FOPI is shown in Fig. 1.

Then, the closed-loop system is described by (5), where

$$E = \begin{bmatrix} 4 & 9 & 9 & -2 & 10 & 7 & 3 & 0 & 0 \\ 1 & 5 & 2 & 2 & 3 & 1 & 1 & 0 & 0 \\ 1 & 0 & -2 & -2 & 6 & 4 & 1 & 0 & 0 \\ 5 & -2 & -3 & 18 & 3 & 16 & 2 & 0 & 0 \\ 6 & 8 & 6 & 8 & 6 & 14 & 2 & 0 & 0 \\ 2 & 11 & 3 & 6 & 6 & 2 & 2 & 0 & 0 \\ 4 & 5 & 5 & 6 & 2 & 9 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \tilde{E} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$A = \begin{bmatrix} -15 & -43 & -39 & 4 & -35 & -22 & -5 & 0 & 0 \\ -3 & -19 & -2 & -5 & -12 & -2 & 1 & 0 & 0 \\ -4 & -16 & -30 & 6 & -9 & -1 & -7 & 0 & 0 \\ -25 & -2 & 3 & -72 & -3 & -74 & -4 & 0 & 0 \\ -27 & -32 & -23 & -39 & -24 & -66 & -5 & 0 & 0 \\ -8 & -41 & -15 & -18 & -24 & -8 & -8 & 0 & 1.0 \\ -18 & -15 & -9 & -29 & -13 & -48 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 10 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 7 & 0 & 1 & -1 \end{bmatrix}.$$

The stability of the closed-loop system can be checked by calculating the eigenvalues of the matrix pencil $sF - G$, where F, G, are defined in (6). In our case:

$$F = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 9 & 9 & -2 & 10 & 7 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 5 & 2 & 2 & 3 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -2 & -2 & 6 & 4 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 & -2 & -3 & 18 & 3 & 16 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 6 & 8 & 6 & 8 & 6 & 14 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 11 & 3 & 6 & 6 & 2 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 5 & 5 & 6 & 2 & 9 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

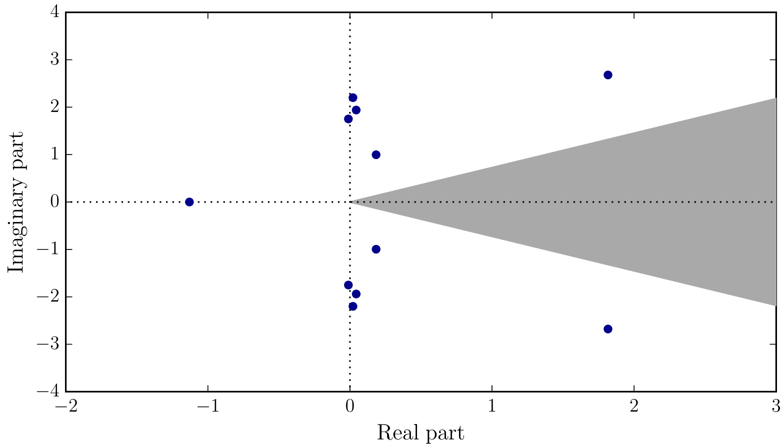

Fig. 2. Numerical example 1. Eigenvalues of $sF - G$. Shaded is the region of instability $|\phi| < 0.628$ rad.

$$G = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ -15 & -43 & -39 & 4 & -35 & -22 & -5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -3 & -19 & -2 & -5 & -12 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -4 & -16 & -30 & 6 & -9 & -1 & -7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -25 & -2 & 3 & -72 & -3 & -74 & -4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -27 & -32 & -23 & -39 & -24 & -66 & -5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -8 & -41 & -15 & -18 & -24 & -8 & -8 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -18 & -15 & -9 & -29 & -13 & -48 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 10 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 7 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

The pencil $sF - G$ has $\hat{p} = 11$ distinct finite eigenvalues $\hat{\lambda}_{1,2} = 1.816 \pm j2.679$, $\hat{\lambda}_3 = -1.130$, $\hat{\lambda}_{4,5} = 1.840 \pm j0.996$, $\hat{\lambda}_{6,7} = -0.0109 \pm j1.751$, $\hat{\lambda}_{8,9} = -0.044 \pm j1.940$, $\hat{\lambda}_{10,11} = 0.0211 \pm j2.2004$, and the infinite eigenvalue $\lambda_{12} = \infty$ with algebraic multiplicity $\hat{q} = 7$. For fractional order $\gamma = 0.6$, we have that $\tilde{\gamma} = \min\{0.6, 0.4\} = 0.4$ in (16), and thus, the closed-loop system is stable if the arguments $\phi_i = \mathrm{Arg}(\hat{\lambda}_i)$ of the finite eigenvalues $\hat{\lambda}_i$, $i = 1, 2, \ldots, 11$, satisfy:

$$|\phi_i| > \frac{\pi}{5} = 0.628 \text{ rad} .$$

The finite eigenvalues of $sF - G$ are illustrated in Fig. 2. The system is stable, since all eigenvalues lie in the region given in . Finally, the general analytical solution of the system requires to calculate the matrices Q_p, J_p (see (15)). These matrices are provided in Appendix A.

3.2. Numerical example 2

Power system models for rotor angle and voltage transient stability studies are formulated as a set of non-linear differential algebraic equations. Transient stability refers to the ability of a power system to maintain synchronism and restore a stationary condition after a perturbation.

The semi-implicit formulation of a power system model is as follows [39]:

$$\begin{bmatrix} T & 0 \\ R & 0 \end{bmatrix} \begin{bmatrix} \dot{x} \\ 0 \end{bmatrix} = \begin{bmatrix} f(x,y) \\ g(x,y) \end{bmatrix},$$

where f, g are the differential, algebraic equations, respectively; $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$ are the state and algebraic variables, respectively; and T, R, are the left handside coefficient matrices. If the examined perturbations are sufficiently small,

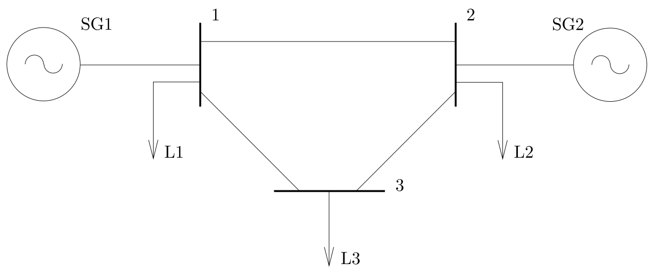

Fig. 3. 3-bus power system scheme.

Table 1
3-bus power system. Branch data with power base $S_b = 100$ MVA.

Branch #	From (i) #	To (k) #	$R_{i,k}$ [pu[a](Ω)]	$X_{i,k}$ [pu(Ω)]	$B_{i,k}$ [pu(Ω^{-1})]
1	1	2	0.022	0.220	0.385
2	1	3	0.010	0.110	0.385
3	2	3	0.011	0.110	0.385

[a]per unit system (pu); in the analysis of power systems, quantities are often expressed as fractions of defined base units.

Table 2
3-bus power system. Data of the synchronous generators.

Generator #	1	2
S_n [MVA]	1800.0	1800.0
V_n [pu(kV)]	230.0	230.0
Model dyn. order	5	6
H [MWs/MVA]	6.500	6.175
T'_{do}, T''_{do} [s]	8.00, 0.03	8.00, 0.03
T'_{qo}, T''_{qo} [s]	0.00, 0.03	0.40, 0.05
X_d, X'_d, X''_d [pu(Ω)]	1.80, 0.30, 0.25	1.80, 0.30, 0.25
X_q, X'_q, X''_q [pu(Ω)]	1.70, 0.55, 0.25	1.70, 0.55, 0.25
X_l [pu(Ω)]	0.20	0.20
R_α [pu(Ω)]	0.0025	0.0025

stability can be assessed by considering the linearized power system model around a stationary point (x_0, y_0). This system can be described as:

$$T\Delta\dot{x} = f_x\Delta x + f_y\Delta y$$
$$R\Delta\dot{x} = g_x\Delta x + g_y\Delta y \ ,$$

where f_x, f_y, g_x, g_y are Jacobian matrices, and $\Delta x = x - x_0$, $\Delta y = y - y_0$. Equivalently, we can write:

$$E_{pr}Y' = A_{pr}Y \ ,$$

where $Y = \begin{bmatrix}\Delta x\\ \Delta y\end{bmatrix}$, $E_{pr} = \begin{bmatrix}T & 0\\ R & 0\end{bmatrix}$, $A_{pr} = \begin{bmatrix}f_x & f_y\\ g_x & g_y\end{bmatrix}$.

The power system considered in this example is shown in Fig. 3. It consists of two synchronous generators (SG), each of which is equipped with an automatic voltage regulator (AVR) that stabilizes the terminal voltage by controlling the current in the rotor field winding; three electric power transmission lines, which operate at nominal voltage 230 kV and nominal frequency 60 Hz; and three constant power loads. The models of the components used in this example can be found in many power system textbooks, e.g. in [40]. Table 1 shows the transmission line parameters. The parameters of the SG and AVR models are shown in Tables 2 and 3, respectively.

The matrices T, R, f_x, f_y, g_x and g_y, which define the linearized system around the examined stationary point, are given in Appendix B. This system is stable, since the real parts of all finite eigenvalues of the pencil $sE_{pr} - A_{pr}$ are negative. In particular, the $sE_{pr} - A_{pr}$ has $p = 19$ finite eigenvalues and the infinite eigenvalue with algebraic multiplicity $q = 32$. The rightmost eigenvalues of $sE_{pr} - A_{pr}$ are shown in Fig. 4. The dominant complex pair of eigenvalues is $-0.181 \pm j4.521$ and represents the electro-mechanical oscillatory mode of the system. This can be found for example by carrying modal participation analysis of the system.

A complex eigenvalue λ_i can be written as $\lambda_i = |\lambda_i|(\cos\phi_i + j\sin\phi_i)$. Then, the damping ratio $\zeta_i = -\cos\phi_i$ measures how the oscillation decays after a perturbation. In a first order system, zero damping defines the border between stability and instability and corresponds to $\phi_m = \frac{\pi}{2}$ rad. The power system is said to be well-damped, if for all oscillatory modes,

Table 3
3-bus power system. Data of the synchronous generator AVRs.

Generator #	1	2
Model dyn. order	4	4
K_α, K_e, K_f	40.0, 1.00, 0.001	40.0, 1.00, 0.001
T_α, T_e, T_f	0.055, 0.36, 1.0	0.055, 0.30, 1.0
T_r	0.05	0.05
A_e, B_e	0.0056, 1.075	0.0056, 1.075
v_α^{min}, v_α^{max}	−5.0 , 5.0	−5.0 , 5.0

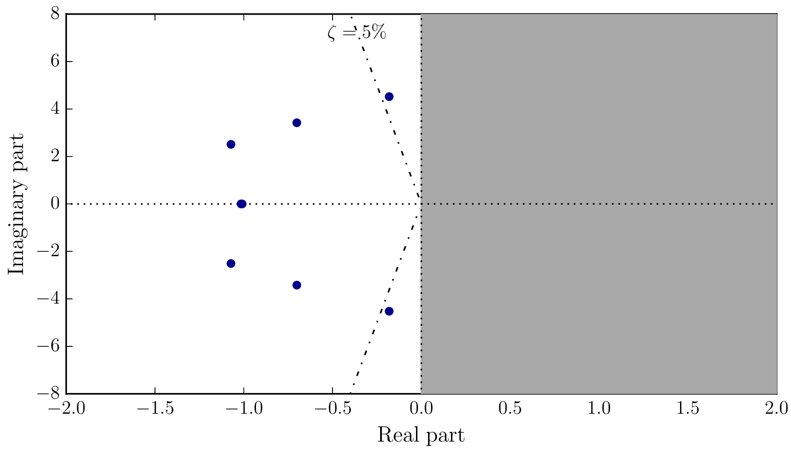

Fig. 4. 3-bus power system. Rightmost eigenvalues of $sE_{pr} - A_{pr}$.

the damping ratio is higher than a threshold, e.g. $\zeta > 5\%$. In Fig. 4, we have drawn the lines that define a damping of $\zeta = 5\%$. As it can be seen, the dominant pair is not well damped.

A standard solution for suppressing a not well damped electro-mechanical oscillation in power systems is the installation of a power system stabilizer (PSS). In this example, we study a fractional order PSS (FOPSS). The FOPSS input is assumed to be the rotor speed of synchronous generator SG1, while the control output is considered to be part of the SG1 AVR reference voltage [40]. That said, the open-loop system takes the form of (2), (3), where: $B_{pr} \in \mathbb{R}^{51\times 1}$, $C_{pr} \in \mathbb{R}^{1\times 51}$; B_{pr} defines the control placement (SG1 AVR reference), and its elements are $B_{pr}(i) = 0$, if $i \neq 50$, $B_{pr}(i) = 1$, if $i = 50$; C_{pr} defines the measured variable (SG1 rotor speed), and its elements are $C_{pr}(i) = 0$, if $i \neq 2$, $C_{pr}(i) = 1$, if $i = 2$; and $D_{pr} = 0$.

The block diagram of the FOPSS is shown in Fig. 5. It consists of a washout filter and three fractional order lead–lag blocks. The FOPSS can be written in the form of (4), where:

$$E_{c1} = \begin{bmatrix} T_w & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ T_w & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad E_{c2} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & T_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & T_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & T_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & T_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & T_6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & T_5 & 0 \end{bmatrix},$$

$$A_c = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & K & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix}, \quad B_c = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad C_c = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}^T.$$

The parameters of the FOPSS are summarized in Table 4. The closed-loop system is calculated according to (5). Finally, we compute the eigenvalues of the pencil $sF - G$, where F, G are defined according to (6). The pencil $sF - G$ has $\hat{p} = 43$ distinct finite eigenvalues and the infinite eigenvalue with algebraic multiplicity $\hat{q} = 77$. For $\gamma = 0.8$, we have that $\tilde{\gamma} = \min\{0.8, 0.2\} = 0.2$ in (16), and thus, the closed-loop system is stable if the arguments $\phi_i = \mathrm{Arg}(\hat{\lambda}_i)$ of the finite

Table 4
FOPSS parameters.

$T_w = 14$ s, $T_1 = T_3 = T_5 = 0.5056$ s, $T_2 = T_4 = T_6 = 0.1007$ s, $K_w = 20$, $\gamma = 0.8$

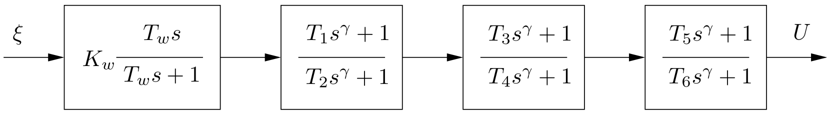

Fig. 5. FOPSS block diagram.

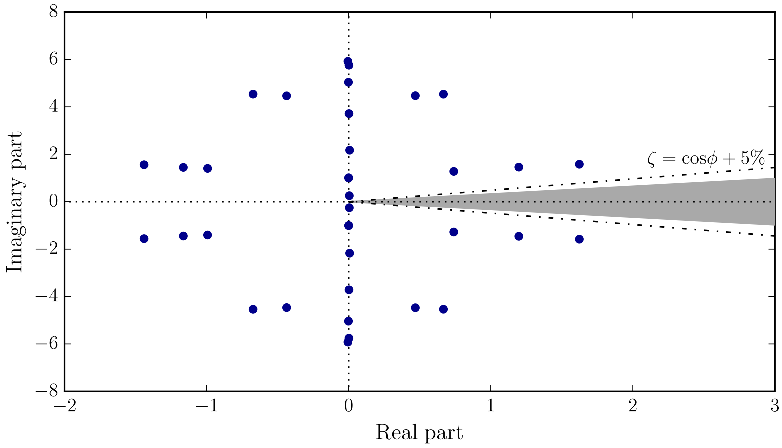

Fig. 6. 3-bus power system. Rightmost eigenvalues of $sF - G$. Shaded is the region of instability $|\phi| < 0.314$ rad.

eigenvalues $\hat{\lambda}_i$, $i = 1, 2, \ldots, 11$, satisfy:

$$|\phi_i| > \tilde{\gamma}\frac{\pi}{2} = \frac{\pi}{10} = 0.314 \text{ rad} .$$

The rightmost eigenvalues of $sF - G$ are shown in Fig. 6. The border between stability and instability is $\phi_m = 0.314$ rad, and this corresponds to a $\zeta_m = -\cos\phi_m = -0.95$. We say that the system is well-damped, if for all eigenvalues it

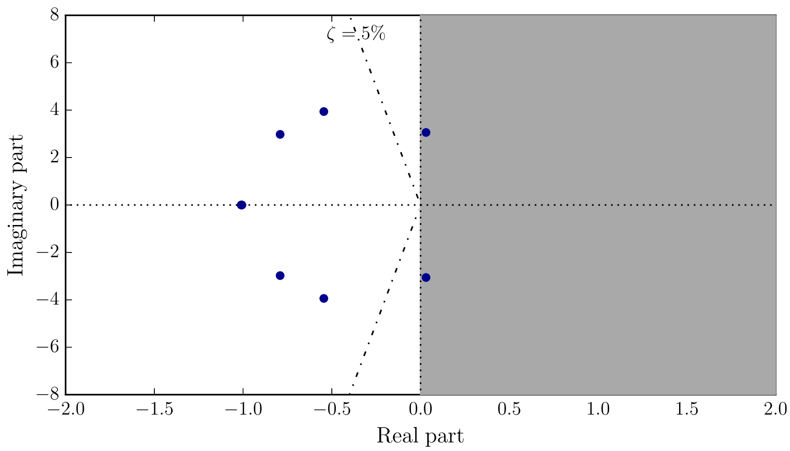

Fig. 7. 3-bus power system. Rightmost eigenvalues of $s\hat{E}_{pr} - \hat{A}_{pr}$.

is $\zeta_i > -\cos\phi_m + 0.05$, that in our case corresponds to $\zeta_i = -0.9$. In Fig. 6, we have drawn the lines that define $\zeta = -\cos\phi_m + 0.05 = -0.9$. As it can be seen, the system is well-damped.

Finally, we provide a simple test on the robustness of the FOPSS. We consider that the transmission line that connects buses 1 and 3 is out of service. In the new stationary point, let $\hat{E}_{pr}$, $\hat{A}_{pr}$, be the matrices that describe the system without the FOPSS. The rightmost eigenvalues of $s\hat{E}_{pr} - \hat{A}_{pr}$ are shown in Fig. 7. The stability of the system with inclusion of the FOPSS can be studied by calculating the new matrices $\hat{F}$, $\hat{G}$. The rightmost eigenvalues of $s\hat{F} - \hat{G}$ are shown in Fig. 8. As it can be seen, the 3-bus system with the line 1–3 out of service is unstable if the FOPSS is not installed. With the FOPSS, the system is stable and well-damped.

4. Conclusions

This article considers the singular linear system of first order (2), and the output (3). We introduce a generalized fractional order feedback controller of (C) type (4), and study the closed loop system (6). The results provide insight on the existence and uniqueness of solutions, as well as on the stability of systems with inclusion of fractional order controllers.

A further extension of this paper is to study practical implementation aspects of fractional order controllers for electrical power system applications, and also provide results on the stability analysis of such dynamic systems with inclusion of fractional order derivatives. We also aim to provide a method to measure the participation of system eigenvalues in system states, and *vice versa*, for singular linear systems of FDEs.

Acknowledgments

This work is supported by the Science Foundation Ireland (SFI), by funding Ioannis Dassios, Georgios Tzounas and Federico Milano, under Investigator Programme Grant No. SFI/15 /IA/3074.

Appendices

This section provides the data used in the numerical examples. In particular, Appendix A includes the data of numerical example 1 and Appendix B includes the data of numerical example 2.

Appendix A

$$Q_p = \begin{bmatrix}
0.07 & 0.07 & -0.26 & 0.14-0.22i & 0.14+0.22i & -0.15-0.37i & -0.15+0.37i & 0.09+0.01i & 0.09-0.01i & -0.07+0.24i & -0.07-0.24i \\
-0.01 & -0.01 & 0.02 & -0.02-0.01i & -0.02+0.01i & 0.04+0.07i & 0.04-0.07i & -0.02+0.04i & -0.02-0.04i & -0.03i & 0.03i \\
0 & 0 & -0.02 & 0.01i & -0.01i & -0.03i & 0.03i & 0.01i & -0.01i & 0.02i & -0.02i \\
0 & 0 & -0.02 & 0.01+0.03i & 0.01-0.03i & 0.01+0.23i & 0.01-0.23i & 0.03-0.03i & 0.03+0.03i & 0.03i & -0.03i \\
0.01 & 0.01 & 0.03 & 0.01+0.07i & 0.01-0.07i & 0 & 0 & 0.06i & -0.06i & -0.02+0.02i & -0.02-0.02i \\
-0.02 & -0.02 & 0.1 & -0.06+0.04i & -0.06-0.04i & 0.04-0.02i & 0.04+0.02i & -0.04+0.01i & -0.04-0.01i & 0.03-0.02i & 0.03+0.02i \\
-0.02 & -0.02 & 0.02 & -0.02+0.08i & -0.02-0.08i & 0.03+0.05i & 0.03-0.05i & -0.03+0.01i & -0.03-0.01i & 0.01-0.1i & 0.01+0.1i \\
-0.05+0.06i & -0.05-0.06i & -0.89 & 0.29+0.62i & 0.29-0.62i & -0.12-0.24i & -0.12+0.24i & 0.03+0.21i & 0.03-0.21i & -0.1-0.16i & -0.1+0.16i \\
-0.22+0.04i & -0.22-0.04i & -0.19 & -0.11+0.89i & -0.11-0.89i & 0.18-0.39i & 0.18+0.39i & -0.26+0.26i & -0.26-0.26i & 0.14-0.31i & 0.14+0.31i \\
0.12+0.18i & 0.12-0.18i & 0.29 & 0.24+0.1i & 0.24-0.1i & 0.65-0.26i & 0.65+0.26i & -0.02+0.17i & -0.02-0.17i & -0.53-0.16i & -0.53+0.16i \\
-0.01-0.02i & -0.01+0.02i & -0.02 & 0.01-0.03i & 0.01+0.03i & -0.13+0.07i & -0.13-0.07i & -0.09-0.04i & -0.09+0.04i & 0.06 & 0.06 \\
-0.01-0.01i & -0.01+0.01i & 0.02 & -0.01 & -0.01 & 0.06 & 0.06 & -0.01-0.01i & -0.01+0.01i & -0.04+0.01i & -0.04-0.01i \\
0.01i & -0.01i & 0.02 & -0.03+0.02i & -0.03-0.02i & -0.4+0.01i & -0.4-0.01i & 0.06+0.05i & 0.06-0.05i & -0.07-0.01i & -0.07+0.01i \\
0.01+0.02i & 0.01-0.02i & -0.04 & -0.07+0.02i & -0.07-0.02i & -0.01 & -0.01 & -0.12+0.01i & -0.12-0.01i & -0.05-0.05i & -0.05+0.05i \\
-0.04-0.07i & -0.04+0.07i & -0.11 & -0.05-0.05i & -0.05+0.05i & 0.04+0.08i & 0.04-0.08i & -0.02-0.08i & -0.02+0.08i & 0.05+0.08i & 0.05-0.08i \\
-0.04-0.05i & -0.04+0.05i & -0.02 & -0.09-0.01i & -0.09+0.01i & -0.09+0.05i & -0.09-0.05i & -0.03-0.05i & -0.03+0.05i & 0.22+0.02i & 0.22-0.02i \\
-0.24-0.03i & -0.24+0.03i & 1 & -0.56+0.4i & -0.56-0.4i & 0.43-0.21i & 0.43+0.21i & -0.41+0.07i & -0.41-0.07i & 0.35-0.22i & 0.35+0.22i \\
-0.49-0.51i & -0.49+0.51i & 0.21 & -0.91+0.06i & -0.91-0.06i & 0.68+0.32i & 0.68-0.32i & -0.52-0.48i & -0.52+0.48i & 0.69+0.31i & 0.69-0.31i
\end{bmatrix},$$

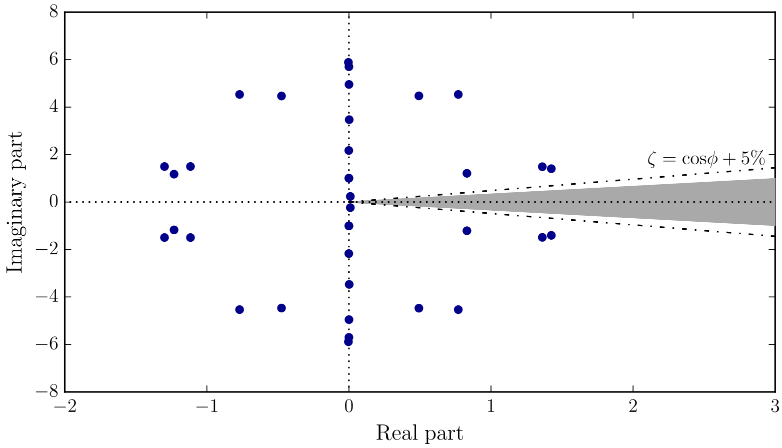

Fig. 8. 3-bus power system. Rightmost eigenvalues of $s\hat{F} - \hat{G}$.

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J_p = \begin{bmatrix}
1.82+2.68i & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1.82-2.68i & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1.13 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0.18+1.0i & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0.18-1.0i & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -0.01+1.75i & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -0.01-1.75i & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.04+1.94i & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.04-1.94i & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.02+2.2i & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.02-2.2i
\end{bmatrix}.
$$

Appendix B

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T = \begin{bmatrix}
0.003 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 234.0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 8.0 & 0.225 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0.03 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0.033 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
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0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.055 & 0 & 0 & 0 \\
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0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.001 & 0 & 0 & 0 & 1.0 & 0 \\
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\end{bmatrix},
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R = \begin{bmatrix}
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f_x = \begin{bmatrix}
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f_y = \begin{bmatrix}
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0 & -5.407 & 3.836 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1.0 & 0 & 0 & 0 & -0.511 & -0.872 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
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g_x = \begin{bmatrix}
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0 & 0 & -0.5 & -0.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
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g_y = \begin{bmatrix}
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0 & 0 & 0 & 0 & 0 & 13.51 & -4.657 & -8.853 & 4.253 & 0.135 & 1.594 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1.0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -4.538 & 13.6 & -9.065 & -1.047 & 2.047 & 0.342 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1.0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -8.425 & -8.819 & 17.24 & -3.107 & -2.083 & -1.971 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
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0 & 0 & 0 & 0 & 0 & 0 & -0.643 & 0 & 0 & 0.776 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1.0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0.791 & 0 & 0 & 0.631 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1.0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1.0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.014 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1.0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.014 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 11.48 & 8.397 & 0.791 & 0.643 & -1.0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -8.397 & 11.48 & 0.643 & -0.791 & 0 & -1.0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1.0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1.0
\end{bmatrix}.
$$

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