A formula of solutions for non-autonomous linear difference equations with a fractional forward operator

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Abstract: In this article we define a fractional forward discrete operator. Then, for a family of linear non-autonomous fractional difference equations constructed by using this fractional discrete operator, we provide a practical formula of solutions. This family of problems covers several linear fractional difference equations that appear in the literature. Numerical examples are given to justify our theory.

Keywords : forward operator, discrete fractional, non-autonomous, numerical, delta.

1 Introduction

Let $Y_k : \mathbb{Z} \to \mathbb{C}^m$ be a vector of sequences. Then the backward difference (or nabla) operator of ν-th order, denoted by ∇^ν, is defined as:

$$\nabla^\nu Y_k = \sum_{j=0}^{\nu} (-1)^j \begin{pmatrix} \nu \\ j \end{pmatrix} Y_{k-j}, \quad \nu \in \mathbb{N}.$$

The ν-th order forward difference operator, denoted by Δ^ν (delta discrete operator), is defined as:

$$\Delta^\nu Y_k = \sum_{j=0}^{\nu} (-1)^{\nu-j} \begin{pmatrix} \nu \\ j \end{pmatrix} Y_{k+j}, \quad \nu \in \mathbb{N}.$$

Both the nabla and the delta operator can then be used to form difference equations that construct models related to macroeconomics, storage, exports and imports, biology, psychology, and time scale analysis, see [1, 2, 3, 4, 5].

In recent years, research has been expanded into studying the nabla difference operator of fractional order, see [6, 7, 8, 9], which is a backward operator. This backward operator has helped in constructing fractional difference equations for the modeling of tumour growth, storage, electricity markets, macroeconomics and logistics, see [10, 11, 12], and mostly aims to extract information from the past.

In this article we will use the *fractional forward* delta operator as defined when applied to a sequence.

The paper is organized as follows. The main results of the paper are presented in Section 2. Section 3 discusses numerical examples that justify our theory. Finally, conclusions are drawn in Section 4.

2 Main Results

In this section we provide our main results. Firstly it is important to understand how the fractional forward delta operator is constructed. We present the following Proposition:

Proposition 2.1. Let $k \in {}_b\mathbb{Z} = \{..., 0, 1, 2, ..., b\}$. If n is a fractional number, then the fractional forward delta operator of n-th order is given by:

$$ {}_b\Delta^{-n} Y_k = \sum_{j=k}^{b} d_{j-k} Y_j, \quad k \in {}_b\mathbb{Z}, \tag{1} $$

where $d_{j-k} = \frac{(j-(k-1))^{\overline{n-1}}}{\Gamma(n)}$, $j = k, k+1, k+2, ..., b$.

Proof. We recall the delta operator of first order $\Delta Y_k = Y_{k+1} - Y_k$ applied to a vector of sequences $Y_k : {}_b\mathbb{Z} \to \mathbb{C}^m$. By using this operator, and by considering a known vector of sequences $f_k : {}_b\mathbb{Z} \to \mathbb{C}^m$, we consider the following system of difference equations of first order:

$$ \Delta Y_k = f_k, \quad Y_b = 0_{m,1}, \quad k \in {}_b\mathbb{Z}. $$

Considering a fixed $N \in {}_b\mathbb{Z}$, we have from the last equation:

$$ \begin{gathered} Y_b - Y_{b-1} = f_{b-1} \\ Y_{b-1} - Y_{b-2} = f_{b-2} \\ \vdots \\ Y_{N+1} - Y_N = f_N. \end{gathered} $$

By taking the sum of the above equations we get $Y_b - Y_N = \sum_{j=N}^{b-1} f_j$. Hence the solution to the system is $Y_k = \sum_{j=k}^{b-1} f_j$. Similarly, if we consider the following system of difference equations of ν-th order:

$$\Delta^{\nu} Y_k = f_k, \quad Y_b = Y_{b-1} = ... = Y_{b-(\nu-1)} = 0_{m,1}, \quad k \in {}_b\mathbb{Z},$$

we receive the solution:

$$Y_k = \sum_{j=k}^{b-1} \frac{(j-(k-1))^{\overline{\nu-1}}}{\Gamma(\nu)} f_j.$$

Where $\Gamma(\cdot)$ is the Gamma function, and $(j-(k-1))^{\overline{\nu-1}} = \frac{\Gamma(j-(k-1)+(\nu-1))}{\Gamma(j-(k-1))} = \frac{\Gamma(j-k+\nu)}{\Gamma(j-k+1)}$. The solution of the above ν-th order system of difference equations can be also expressed as:

$$Y_k = \Delta^{-\nu} f_k.$$

Hence we have that:

$$\Delta^{-\nu} f_k = \sum_{j=k}^{b-1} \frac{(j-(k-1))^{\overline{\nu-1}}}{\Gamma(\nu)} f_j.$$

Based on this expression, if n is a fractional number, then the fractional forward delta operator of n-th order is defined as:

$${}_{b-1}\Delta^{-n} Y_k = \sum_{j=k}^{b-1} d_{j-k} Y_j, \quad k \in {}_{b-1}\mathbb{Z},$$

or, equivalently:

$${}_b\Delta^{-n} Y_k = \sum_{j=k}^{b} d_{j-k} Y_j, \quad k \in {}_b\mathbb{Z},$$

where $d_{j-k} = \frac{(j-(k-1))^{\overline{n-1}}}{\Gamma(n)}$, $j = k, k+1, k+2, ..., b$. The proof is completed.

One important difference between the fractional forward delta operator of n-th order (n fractional) and the forward delta operator of ν-th order (ν natural) is that when Δ^{ν} is applied to Y_k we receive an expression that includes Y_k, Y_{k+1}, Y_{k+2}, ..., $Y_{k+\nu}$ whereas when ${}_b\Delta^{-n}$ is applied to Y_k we receive an expression that includes Y_k, Y_{k+1}, Y_{k+2}, ..., Y_b.

Consider the following linear system of non-autonomous difference equations of fractional order:

$$E_k \, {}_b\Delta^{-n} Y_k = \sum_{j=k}^{b} A_k^{(j)} Y_j + V_k, \quad k \in {}_b\mathbb{Z}. \tag{2}$$

Where $Y_k : {}_b\mathbb{Z} \to \mathbb{C}^m$, and $E_k, A_k^{(j)} : {}_b\mathbb{Z} \to \mathbb{C}^{r\times m}$, $V_k : {}_b\mathbb{Z} \to \mathbb{C}^r$. The symbol (j) on the matrix $A_k^{(j)}$ refers to the fact that this matrix is a coefficient of Y_j in (2). For simplicity, in this article we will focus on the case that $r = m = 1$, i.e. instead of a system we have a generalized non-autonomous difference equation, and provide a practical formula of its solutions. We prove the following theorem:

Theorem 2.1. Consider (2) for $r = m = 1$. Then if $\forall k \in {}_b\mathbb{Z}$ and $E_k \neq A_k^{(k)}$, there always exists a unique solution for (2) given by:

$$Y_k = (-1)^{b-k}\frac{\mathcal{D}_{k,b}}{\Pi_{i=k}^{b-1}D_i}C + \sum_{j=k}^{b-1}(-1)^{j-k}\frac{\mathcal{D}_{k,j}}{\Pi_{i=k}^{j}D_i}V_j, \quad k \in {}_b\mathbb{Z}. \tag{3}$$

Where $\mathcal{D}_{k,k} = 1$, and for $j \neq k$, $\mathcal{D}_{k,j}$ is the determinant of a $(j-k)\times(j-k)$ matrix:

$$\mathcal{D}_{k,j} = \begin{vmatrix} D_{k,k+1} & D_{k,k+2} & D_{k,k+3} & \cdots & D_{k,j-1} & D_{k,j} \\ D_{k+1} & D_{k+1,k+2} & D_{k+1,k+3} & \cdots & D_{k+1,j-1} & D_{k+1,j} \\ 0 & D_{k+2} & D_{k+2,k+3} & \cdots & D_{k+2,j-1} & D_{k+2,j} \\ 0 & 0 & D_{k+3} & \cdots & D_{k+3,j-1} & D_{k+3,j} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & D_{j-1} & D_{j-1,j} \end{vmatrix}. \tag{4}$$

In addition C is constant, and:

$$D_{k,j} = d_{j-k}E_k - A_k^{(j)}, \quad d_{j-k} = \frac{(j-(k-1))^{\overline{n-1}}}{\Gamma(n)}, \quad D_j = D_{j,j}. \tag{5}$$

Proof. By replacing (1), i.e. ${}_b\Delta^{-n}Y_k = \sum_{j=k}^{b} d_{j-k}Y_j$, into (2) we get:

$$E_k\sum_{j=k}^{b} d_{j-k}Y_j = \sum_{j=k}^{b} A_k^{(j)}Y_j + V_k, \quad k \in {}_b\mathbb{Z},$$

or, equivalently:

$$\textstyle\sum_{j=k}^{b}(d_{j-k}E_k - A_k^{(j)})Y_j = V_k,$$

whereby setting $D_{k,j} = d_{j-k}E_k - A_k^{(j)}$, and $D_j = D_{j,j}$:

$$\sum_{j=k}^{b} D_{k,j}Y_j = V_k,$$

with

$$Y_b = \frac{1}{D_b} V_b. \tag{6}$$

It is easy to observe that $Y_{b-1} = \frac{1}{D_{b-1}} V_{b-1} - \frac{D_{b-1,b}}{D_{b-1} D_b} V_b$. We will replace k with $k = b-2, b-3, ..., b-m$, $m \in {}_b\mathbb{Z}$. For $k = b-2$:

$$Y_{b-2} = \frac{1}{D_{b-2}} V_{b-2} - \frac{D_{b-2,b-1}}{D_{b-2} D_{b-1}} V_{b-1} + \frac{\begin{vmatrix} D_{b-2,b-1} & D_{b-2,b} \\ D_{b-1} & D_{b-1,b} \end{vmatrix}}{D_{b-2} D_{b-1} D_b} V_b.$$

Whereby using mathematical induction for $k = b-m$ and $m \in {}_b\mathbb{Z}$, we arrive at:

$$Y_{b-m} = \frac{1}{D_{b-m}} V_{b-m} - \frac{D_{b-m,b-(m-1)}}{D_{b-m} D_{b-(m-1)}} V_{b-(m-1)} + \ldots (-1)^m \frac{\mathcal{D}_{b-m,b}}{D_{b-m} \ldots D_{b-1} D_b} V_b.$$

Where

$$\mathcal{D}_{b-m,b} = \begin{vmatrix} D_{b-m,b-(m-1)} & D_{b-m,b-(m-2)} & D_{b-m,b-(m-3)} & \cdots & D_{b-m,b-1} & D_{b-m,b} \\ D_{b-(m-1)} & D_{b-(m-1),b-(m-2)} & D_{b-(m-1),b-(m-3)} & \cdots & D_{b-(m-1),b-1} & D_{b-(m-1),b} \\ 0 & D_{b-(m-2)} & D_{b-(m-2),b-(m-3)} & \cdots & D_{b-(m-2),b-1} & D_{b-(m-2),b} \\ 0 & 0 & D_{b-(m-3)} & \cdots & D_{b-(m-3),b-1} & D_{b-(m-3),b} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & D_{b-1} & D_{b-1,b} \end{vmatrix}.$$

Equivalently:

$$Y_{b-m} = \sum_{j=b-m}^{b} (-1)^{j-(b-m)} \frac{\mathcal{D}_{b-m,j}}{\Pi_{i=b-m}^{j} D_i} V_j.$$

Where $\mathcal{D}_{b-m,b-m} = 1$, and for $j \neq b-m$:

$$\mathcal{D}_{b-m,j} = \begin{vmatrix} D_{b-m,b-(m-1)} & D_{b-m,b-(m-2)} & D_{b-m,b-(m-3)} & \cdots & D_{b-m,j-1} & D_{b-m,j} \\ D_{b-(m-1)} & D_{b-(m-1),b-(m-2)} & D_{b-(m-1),b-(m-3)} & \cdots & D_{b-(m-1),j-1} & D_{b-(m-1),j} \\ 0 & D_{b-(m-2)} & D_{b-(m-2),b-(m-3)} & \cdots & D_{b-(m-2),j-1} & D_{b-(m-2),j} \\ 0 & 0 & D_{b-(m-3)} & \cdots & D_{b-(m-3),j-1} & D_{b-(m-3),j} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & D_{j-1} & D_{j-1,j} \end{vmatrix},$$

and hence:

$$Y_k = \sum_{j=k}^{b} (-1)^{j-k} \frac{\mathcal{D}_{k,j}}{\Pi_{i=k}^{j} D_i} V_j.$$

Where $\mathcal{D}_{k,j}$ is defined by (4). Equivalently:

$$Y_k = (-1)^{b-k}\frac{\mathcal{D}_{k,b}}{\Pi_{i=k}^{b} D_i}V_b + \sum_{j=k}^{b-1}(-1)^{j-k}\frac{\mathcal{D}_{k,j}}{\Pi_{i=k}^{j} D_i}V_j,$$

or, equivalently, by using (6):

$$Y_k = (-1)^{b-k}\frac{\mathcal{D}_{k,b}}{\Pi_{i=k}^{b} D_i}Y_b D_b + \sum_{j=k}^{b-1}(-1)^{j-k}\frac{\mathcal{D}_{k,j}}{\Pi_{i=k}^{j} D_i}V_j,$$

or, equivalently, and by setting $Y_b = C$, C constant:

$$Y_k = (-1)^{b-k}\frac{\mathcal{D}_{k,b}}{\Pi_{i=k}^{b-1} D_i}C + \sum_{j=k}^{b-1}(-1)^{j-k}\frac{\mathcal{D}_{k,j}}{\Pi_{i=k}^{j} D_i}V_j.$$

The proof is completed.

Remark 2.1. If the condition Y_b is known, then the solution is unique and is given by:

$$Y_k = (-1)^{b-k}\frac{\mathcal{D}_{k,b}}{\Pi_{i=k}^{b-1} D_i}Y_b + \sum_{j=k}^{b-1}(-1)^{j-k}\frac{\mathcal{D}_{k,j}}{\Pi_{i=k}^{j} D_i}V_j.$$

Remark 2.2. We consider (2) for $r = m = 1$, and $V_k = 0$. Then (2) is homogeneous and:

$$Y_k = (-1)^{b-k}\frac{\mathcal{D}_{k,b}}{\Pi_{i=k}^{b-1} D_i}C, \quad k \in {}_b\mathbb{Z}.$$

Remark 2.3. For (2) and $r = m = 1$ the general solution can be written as:

$$Y_k = Y_k^{(h)} + Y_k^{(p)}, \quad k \in {}_b\mathbb{Z},$$

with

$$Y_k^{(h)} = (-1)^{b-k}\frac{\mathcal{D}_{k,b}}{\Pi_{i=k}^{b-1} D_i}C, \quad Y_k^{(p)} = \sum_{j=k}^{b-1}(-1)^{j-k}\frac{\mathcal{D}_{k,j}}{\Pi_{i=k}^{j} D_i}V_j.$$

Where $Y_k^{(h)}$ is the solution of the homogeneous equation $E_k\, {}_b\Delta^{-n} Y_k = \sum_{j=k}^{b} A_k^{(j)} Y_j$, and $Y_k^{(p)}$ a partial solution of (2).

3 Numerical Examples

In this section we present illustrative examples that justify our theory.

Example 3.1.

We consider the following fractional equation:

$$ {}_5\Delta^{-0.5} Y_k = AY_k + k^3, \quad k \in {}_5\mathbb{Z}, $$

with $Y_k : {}_5\mathbb{Z} \to \mathbb{C}$ and $A \in \mathbb{C}$ with $A \neq 1$. From (3) the solution is given by:

$$ Y_k = (-1)^{5-k} \frac{\mathcal{D}_{k,5}}{\Pi_{i=k}^{4} D_i} C + \sum_{j=k}^{4} (-1)^{j-k} \frac{\mathcal{D}_{k,j}}{\Pi_{i=k}^{j} D_i} j^3, \quad k \in {}_5\mathbb{Z}. $$

This is a practical formula of solutions. For example at $k = 3$ we get:

$$ Y_3 = \frac{\mathcal{D}_{3,5}}{\Pi_{i=3}^{4} D_i} C + \sum_{j=3}^{4} (-1)^{j-3} \frac{\mathcal{D}_{3,j}}{\Pi_{i=3}^{j} D_i} j^3. $$

Where $D_{3,j} = \frac{(j-2)^{\overline{-0.5}}}{\Gamma(0.5)}$ for $j \neq 3$, and $D_j = \frac{(1)^{\overline{-0.5}}}{\Gamma(0.5)} - A = 1 - A$. In addition:

$$ \mathcal{D}_{3,5} = \begin{vmatrix} D_{3,4} & D_{3,5} \\ D_4 & D_{4,5} \end{vmatrix} = \begin{vmatrix} \frac{(2)^{\overline{-0.5}}}{\Gamma(0.5)} & \frac{(3)^{\overline{-0.5}}}{\Gamma(0.5)} \\ 1 - A & \frac{(2)^{\overline{-0.5}}}{\Gamma(0.5)} \end{vmatrix} = \begin{vmatrix} 0.5 & \frac{3}{8} \\ 1 - A & 0.5 \end{vmatrix} = \frac{3A - 1}{8}, $$

and

$$ \mathcal{D}_{3,4} = D_{3,4} = \frac{(2)^{\overline{-0.5}}}{\Gamma(0.5)} = 0.5, \quad \mathcal{D}_{3,3} = 1. $$

Note that $\frac{(2)^{\overline{-0.5}}}{\Gamma(0.5)} = \frac{\Gamma(1.5)}{\Gamma(2)\Gamma(0.5)} = \frac{\Gamma(1.5)}{2\Gamma(1.5)} = \frac{1}{2}$, $\frac{(3)^{\overline{-0.5}}}{\Gamma(0.5)} = \frac{\Gamma(2.5)}{\Gamma(3)\Gamma(0.5)} = \frac{\Gamma(2.5)}{2\frac{4}{3}\Gamma(2.5)} = \frac{3}{8}$.
Thus:

$$ Y_3 = \frac{3}{8(1-A)^2} C + \frac{27}{1-A} - \frac{32}{(1-A)^2}. $$

In Figure 1 we see Y_3 for $C, A \in [-400, 400]$. In Figure 2 we set the condition $Y_5 = C = 200$ and see Y_3 as A changes.

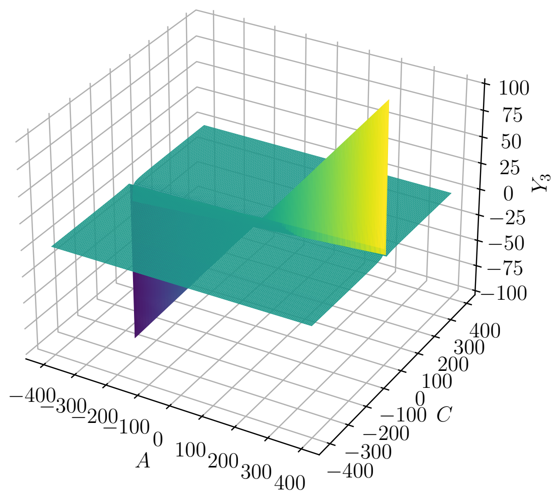

Figure 1: A vs C vs Y_3, Example 3.1.

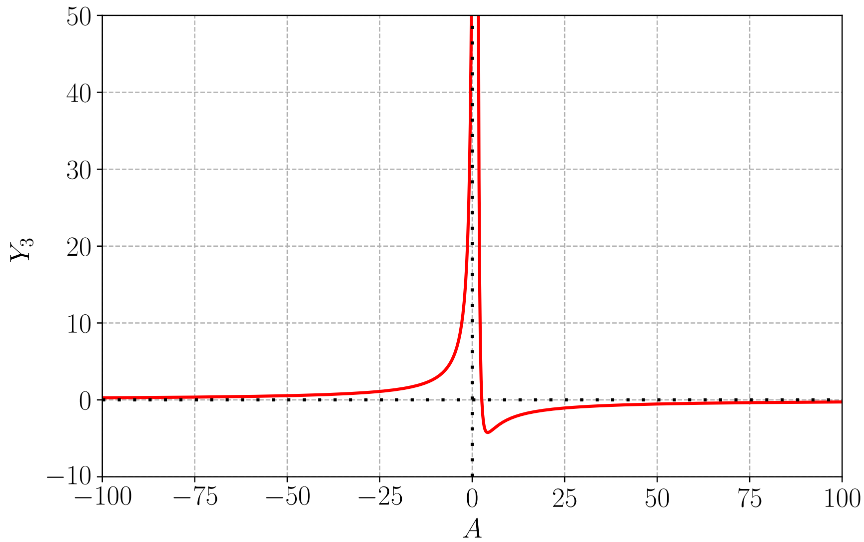

Figure 2: A versus Y_{10} for $C = 200$, Example 3.1.

Example 3.2.

We consider the following fractional equation:

$$k\ {}_{10}\Delta^{-0.5}Y_k = (k - \underset{8}{2})Y_k + (k-7)^3 Y_{k+1},$$

with $Y_k : {}_{10}\mathbb{Z} \to \mathbb{C}$. Its solution is given by (3):

$$Y_k = (-1)^{10-k} \frac{\mathcal{D}_{k,10}}{\Pi_{i=k}^{9} D_i} C, \quad k \in {}_{10}\mathbb{Z}.$$

At $k = 7$ we get:

$$Y_7 = -\frac{\mathcal{D}_{7,10}}{\Pi_{i=7}^{9} D_i} C$$

From (5) we have $d_0 = 1$, $d_1 = 0.5$, $d_2 = \frac{3}{8}$, $d_3 = \frac{5}{16}$, $D_7 = D_8 = D_9 = 2$, and:

$$\begin{gathered} D_{7,8} = d_1 E_7 - A_7^{(8)} = \tfrac{7}{2} \\ D_{7,9} = d_2 E_7 - A_7^{(9)} = \tfrac{21}{8} \\ D_{7,10} = d_3 E_7 - A_7^{(10)} = \tfrac{35}{16} \\ D_{8,9} = d_1 E_8 - A_8^{(9)} = 3 \\ D_{8,10} = d_2 E_8 - A_8^{(10)} = 3 \\ D_{9,10} = d_1 E_9 - A_9^{(10)} = -\tfrac{7}{2} \end{gathered}$$

In addition:

$$\mathcal{D}_{7,10} = \begin{vmatrix} D_{7,8} & D_{7,9} & D_{7,10} \\ D_8 & D_{8,9} & D_{8,10} \\ 0 & D_9 & D_{9,10} \end{vmatrix} = \begin{vmatrix} \frac{7}{2} & \frac{21}{8} & \frac{35}{16} \\ 2 & 3 & 3 \\ 0 & 2 & -\frac{7}{2} \end{vmatrix} = -\frac{245}{8},$$

Thus

$$Y_7 = \frac{245}{64} C.$$

Example 3.3.

We consider the following fractional equation:

$${}_{12}\Delta^{-0.5} Y_k = A^k,$$

with $Y_k : {}_{12}\mathbb{Z} \to \mathbb{C}$ and $A \in \mathbb{R}$ with $A \neq 1$. From (3) the solution is given by:

$$Y_k = (-1)^{12-k} \frac{\mathcal{D}_{k,12}}{\Pi_{i=k}^{11} D_i} C + \sum_{j=k}^{11} (-1)^{j-k} \frac{\mathcal{D}_{k,j}}{\Pi_{i=k}^{j} D_i} A^j, \quad k \in {}_{12}\mathbb{Z}.$$

At $k = 10$ we get:

$$Y_{10} = \frac{\mathcal{D}_{10,12}}{\Pi_{i=10}^{11} D_i} C + \sum_{j=10}^{11} (-1)^{j-10} \frac{\mathcal{D}_{10,j}}{\Pi_{i=10}^{j} D_i} A^j.$$

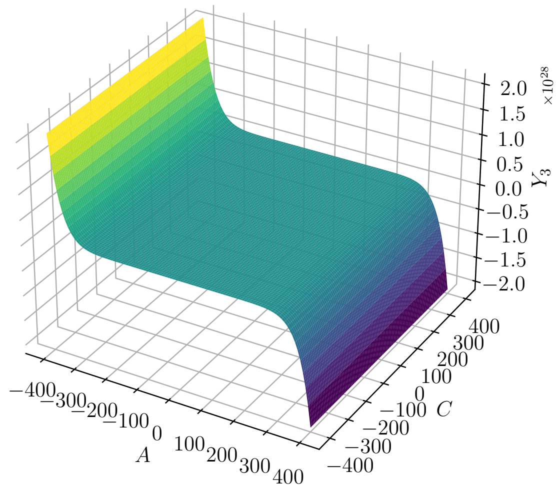

Figure 3: A vs C vs Y_{10}, Example 3.3.

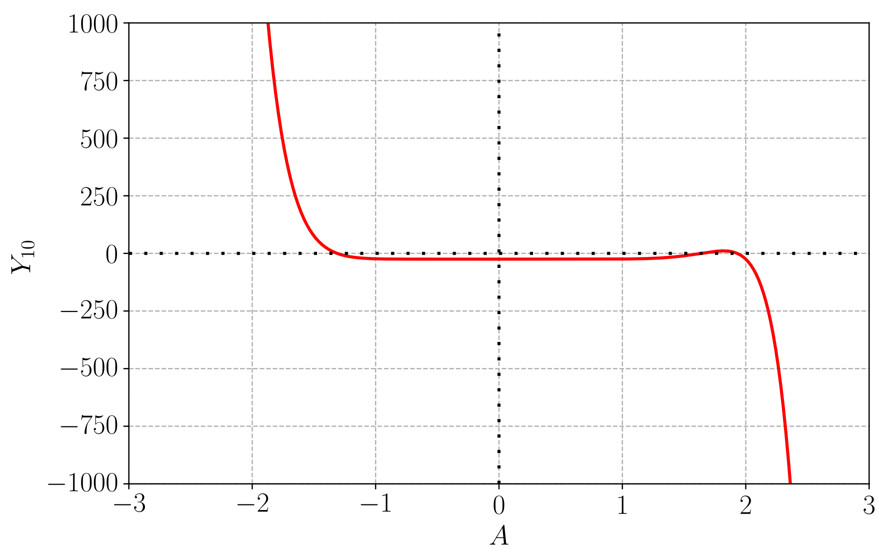

Figure 4: A versus Y_{10} for $C = 200$, Example 3.3.

From (5) we have $d_0 = 1$, $d_1 = 0.5$, $d_2 = \frac{3}{8}$, and $D_{10,11} = D_{11,12} = 0.5$, $D_{10,12} = \frac{3}{8}$, $D_{10} = D_{11} = 1$. In addition:

$$\mathcal{D}_{10,12} = \begin{vmatrix} D_{10,11} & D_{10,12} \\ D_{11} & D_{11,12} \end{vmatrix} = \begin{vmatrix} 0.5 & \frac{3}{8} \\ 1 & 0.5 \end{vmatrix} = -\frac{1}{8}.$$

Thus:

$$Y_{10} = -\frac{1}{8}C + A^{10} - 0.5A^{11}.$$

In Figure 3 we see Y_{10} for $C, A \in [-400, 400]$. In Figure 4 we set the condition $Y_{12} = C = 200$ and see Y_{10} as A changes.

4 Concluding remarks

We studied a class of non-autonomous linear fractional difference equations which are constructed by using a fractional delta-forward discrete operator. We obtained a practical formula of solutions and provided numerical examples to justify our theory.

As a further extension of this article we aim to use this fractional operator to construct a new mathematical model for electricity markets which will provide further insights in decision making. We also aim to extend the results in [13] and construct a fractional forward operator for non-causal systems of differential equations. With this regard, there is already some research in progress.

Data availability statement

The authors do not have any data related to this article.

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Conflict of interest

This work does not have any conflicts of interest.

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