Frequency Divider as a Continuum

Georgios Tzounas, *Member, IEEE*, Ioannis Dassios, and Federico Milano, *Fellow, IEEE*

***Abstract*—The letter describes a novel, continuum-based approach, to capture the evolution of electromechanical dynamics in a power network following a disturbance. Such approach is based on the *frequency divider formula (FDF)*, which was recently proposed by the third author. A key point in obtaining the frequency divider formula (FDF) as a consequence of a continuum, is to show that the spatial rate of change, at a given time, of the frequency along a lossless line is constant. The proposed derivation is then compared with the *electromechanical wave approach (EWA)*, which has been discussed in the literature in a variety of hues and is aimed at modeling the propagation in a power system of frequency oscillations following a disturbance. The discussion illustrates similarities and differences between FDF and electromechanical wave approach (EWA).**

***Index Terms*—Electromechanical dynamics, continuum, frequency divider, electromechanical wave theory.**

I. Introduction

THE letter deals with an important problem in power engineering, i.e. how to accurately capture electromechanical dynamics along transmission lines following a disturbance in the system. A well-known approach to this problem is to treat the power system as a distributed continuum [1], [2], [3], [4], [5], [6], [7].

Modeling power system electromechanical dynamics as a continuum was first proposed in [1]. Therein, Semlyen introduced the *electromechanical wave* theory, according to which, distributing the parameters of lines and generators as a continuum allows representing angle and frequency disturbances as traveling waves. The EWA was fostered by the development of phasor measurement units (PMUs) in the 1990 s, when tests of synchronized PMUs observed cases where devices remote to the disturbance location appeared to respond with a significant delay [2]. In the EWA, this response is linked to the disturbance's travel speed, which is estimated to be much lower than the speed of light.

Manuscript received 9 March 2022; revised 9 May 2022 and 14 July 2022; accepted 11 September 2022. Date of publication 15 September 2022; date of current version 20 October 2022. The work of Georgios Tzounas was supported by the Swiss National Science Foundation through NCCR Automation under Grant 51NF40 18054. The work of Ioannis Dassios and Federico Milano was supported by the Sustainable Energy Authority of Ireland (SEAI), through Project FRESLIPS, under Grant RDD/00681. The work of Federico Milano was also supported by European Commission through Project EdgeFLEX, under Grant 883710. Paper no. PESL-00063-2022. *(Corresponding author: Georgios Tzounas.)*

Georgios Tzounas is with the ETH Zürich, 8092 Zürich, Switzerland (e-mail: georgios.tzounas@eeh.ee.ethz.ch).

Ioannis Dassios and Federico Milano are with the School of Electrical & Electronic Engineering, University College Dublin, Belfield Campus, D04V1W8 Dublin, Ireland (e-mail: ioannis.dassios@ucd.ie; federico.milano@ieee.org).

Digital Object Identifier 10.1109/TPWRS.2022.3207058

Being the focus on electromechanical dynamics, network electromagnetic transients can be assumed to be in *quasi-steady state*, as they are much faster than the time-scale of interest. Under this assumption, several authors have worked on different aspects of the EWA. Some studies have focused on removing a part of assumptions of [1], thus resulting in more complex models that allow for anisotropy, non-homogeneity, and losses in the system, e.g. [3]. Recent works have further investigated properties of the EWA and contributed to the definition of relevant protection and control schemes, e.g. [4], [5], [6], [7].

This letter revisits the idea of modelling power systems as a continuum, and shows that the frequency divider formula (FDF) – see [8] for its theoretical foundation and [9] for a hardware-in-the-loop validation – is in effect a consequence of a continuum-based model. In contrast to EWA, the FDF continuum does not require to distribute the synchronous machines (SMs) but works directly on the original system. This has significant consequences, as shown below, on the interpretation of the dynamic behavior of the system and the propagation of electromechanical oscillations.

The novel contributions of the paper are as follows:

- It is shown that the spatial distribution of frequency variations along a transmission line is linear. As a consequence of the above linearity, the FDF – first proposed to estimate bus frequencies – is formulated as a continuum model that captures the frequency variations along transmission lines.
- A comprehensive comparison between the EWA and FDF-based continuum is provided which features the several advantages that the second has over the first. In particular, it is shown that the EWA leads to inconsistent results, namely, fast dynamics that cannot originate from the actual electromechanical models of synchronous machines.

II. Frequency Divider-Based Continuum Model

The active power flow between two connected buses in a power network, say from bus h to bus k, is as follows:

$$p_{hk} = v_h v_k \left[G_{hk} \cos(\theta_h - \theta_k) + B_{hk} \sin(\theta_h - \theta_k) \right] \, , \quad (1)$$

where $G_{hk} + \jmath B_{hk}$ is the element (h,k) of the grid admittance matrix and $v_i \angle \theta_i$ is the voltage phasor at bus $i = \{h,k\}$.

Let us assume a lossless connection (i.e., $G_{hk} \approx 0$), negligible voltage bus voltage magnitude variations (i.e., $v_h \approx v_k \approx \text{const. pu}$), and that the voltage phase angle difference is small (i.e., $\sin(\theta_h - \theta_k) \approx \theta_h - \theta_k$). These assumptions lead to rewrite (1) as follows:

$$\tilde{p}_{hk} \approx v_h v_k \, \frac{\tilde{\theta}_h - \tilde{\theta}_k}{X} \, , \quad (2)$$

where $X = 1/B_{hk}$ and $\sim$ denotes variation of a quantity with respect to a known operating condition, i.e. $\tilde{y} = y - y_o$. Differentiation of (2) with respect to time gives:

$$\frac{d\tilde{p}}{dt} = -\omega_b v_h v_k \frac{\tilde{\omega}_h - \tilde{\omega}_k}{X} = -\gamma \frac{\tilde{\omega}_h - \tilde{\omega}_k}{X}, \tag{3}$$

where ω_b is the nominal frequency in rad/s; $\gamma = \omega_b v_h v_k$; $\tilde{p} = -\tilde{p}_{hk}$ is taken with negative sign for consistency with the notation utilized in [1]; and $\tilde{\omega}_i$ is the frequency variation in pu at bus $i = \{h, k\}$. The equations so far are written considering a lumped model of the line. Assuming a homogeneous line, one can consider an infinitesimal section of the line conductor with reactance $\ell dX = X dx$, with ℓ being the length of the line. Since there are no losses, the variation $d\tilde{p}$ of active power along dx is zero:

$$\frac{\partial \tilde{p}}{\partial x} = 0, \tag{4}$$

which means that the active power is only a function of time $\tilde{p}(x,t) = \tilde{p}(t)$. (4) implies that, at any given time t, $\partial\tilde{p}/\partial t$ is the same at every point of the line. Then, if $\partial\tilde{\omega}$ denotes the frequency variation along the infinitesimal section ∂x of the line, we get that the spatial rate of change of the frequency $\partial\tilde{\omega}/\partial x = \mathrm{const.}$, and thus the following relationship holds:

$$\frac{\tilde{\omega}_h - \tilde{\omega}_k}{l} = \frac{\partial \tilde{\omega}}{\partial x}.$$

Then, (3) can be rewritten in terms of partial derivatives, as follows:

$$\frac{\partial \tilde{p}}{\partial t} = -\kappa \frac{\partial \tilde{\omega}}{\partial x}, \tag{5}$$

where $\kappa = \gamma\,\ell/X$. The result above indicates that the assumption made in [8] that the frequency distributes linearly along a line is true if the line is lossless.

The same conclusion on the distribution of the frequency can be obtained also by removing the hypothesis of lossless line. Assuming an homogeneous lossy conductor with constant cross section, one has:

$$\frac{\partial \tilde{p}}{\partial x} = \frac{\rho\,\imath^2}{A}, \tag{6}$$

where A and ρ are the cross-section and the resistivity of the line, respectively; and $\imath^2$ is the square of the current injection imposed in the line. If there is no dispersion, the current is the same in every point of the line at any given t. Hence, the right-hand side of (6) depends only on time and the solution of the partial differential (6) has the form:

$$\tilde{p}(x,t) = \tilde{p}_1(x) + \tilde{p}_2(t), \tag{7}$$

where $\tilde{p}_1(x) = (\rho\,\imath^2/A)\,x$ and $\tilde{p}_2(t)$ is an arbitrary function of time. That is, the active power is the sum of two functions, one dependent only on the position and the other only on time. Hence, we also have that, at a given time, the derivative $\partial\tilde{p}(x,t)/\partial t = \partial\tilde{p}_2(t)/\partial t$ is that same at every point of the line. This confirms, also for lossy lines, the main conclusion of the FDF, namely that $\partial\tilde{\omega}/\partial x = \mathrm{const.}$ along a transmission line at any given time t.

Finally, in [8], the SM rotor speed ω_{r} is linked to the bus frequency through the classical model, i.e.:

$$\frac{\partial \tilde{p}}{\partial t} = -\frac{\tilde{\omega}_{\mathrm{r}} - \tilde{\omega}_h}{X'_{\mathrm{d}}}, \tag{8}$$

where X'_{d} is the lumped transient reactance behind the emf e'_q of the machine connected to bus h, and where it is assumed that $e'_q \approx v_h \approx 1$ [8]. Losses are ignored. With a proper coefficient κ, (8) can be rewritten as a continuum in the form of (5).

III. Electromechanical Wave Approach

The original EWA-based continuum power system model, presented in [1], considers a distributed classical machine model with $X'_{\mathrm{d}} = 0$ in a homogeneous, lossless, and radial system with constant voltages. Under these assumptions, and using the notation described in the previous section, the rate of change of the active power along the line becomes:

$$\frac{\partial \tilde{p}}{\partial x} = -m \frac{\partial \tilde{\omega}_{\mathrm{r}}}{\partial t} = -m \frac{\partial \tilde{\omega}}{\partial t}, \tag{9}$$

where $m = M/\ell$, with M being the starting time of the lumped SM[1] and $\tilde{\omega}_{\mathrm{r}} = \tilde{\omega}$ because $X'_{\mathrm{d}} = 0$. Combining (5) and (9), the following expression can be obtained:

$$\frac{\partial^2 \tilde{\omega}}{\partial x^2} = \frac{m}{\kappa} \frac{\partial^2 \tilde{\omega}}{\partial t^2}, \tag{10}$$

which describes a wave with travel speed [1]:

$$c = \sqrt{\frac{\kappa}{m}} = \ell \sqrt{\frac{v_h v_k}{X} \frac{\omega_b}{M}}, \tag{11}$$

where $v_h v_k / X$ is in pu and ω_b/M has the units of s^{-2}.

IV. FDF-Based Continuum vs EWA

The expressions of $\partial\tilde{p}/\partial x$ given by (9) and (4) indicate a conceptual difference between the two approaches. That is, as opposed to the EWA, the FDF continuum does not require to distribute the inertia of SMs. It works directly on the original topology of the system using the standard power equations. This point has several relevant consequences.

In the first place, we note that the accuracy of the FDF is largely independent from the location of SMs. On the other hand, the accuracy of the EWA is known to be highly topology-dependent. In particular, the EWA should be expected to show good accuracy only in part of the grid where a large number of small-capacity SMs are located very close to each other, to an extent that they can be reasonably approximated with a continuum [2]. Large distances between generators, which are common in real-world networks, combined with a rough distribution of SM parameters, result to an EWA continuum model of compromised fidelity. A simple example that illustrates this issue is discussed in Section V-A.

[1] M can be obtained as the sum of the starting times of the machines distributed along the line. Note that the distribution of the machines does not represent an actual topology but is fictitious; it serves to facilitate the application of the EWA to the examined system.

As a second remark, we observe that the implementation of the EWA comes with a need for extensive model simplification. For example, most works based on the EWA assume that SMs are represented using a classical second-order SM model. Moreover, the family of EWAs is intended for SM-dominated power systems. On the other hand, the FDF can handle higher-order machine models and can be generalized to take into account any device connected to the network (and, in turn, any boundary condition imposed to the partial differential equations that describe the continuum), including converter-based resources [10].

It is also relevant to note that, in [1], it is assumed that $\partial\tilde{\omega}/\partial x = \text{const.}$, as opposed to deriving it as we have done in Section II. In [1], assuming linear variation of the frequency is required to impose the following identity:

$$\frac{\ell}{X}\,\frac{\partial \tilde{\omega}}{\partial x} = \frac{\partial \tilde{\omega}}{\partial X} \equiv \frac{\tilde{\omega}_h - \tilde{\omega}_k}{X}\,. \tag{12}$$

However, imposing (12) is not satisfied by the wave (10). This point is further illustrated in Section V.

Finally, from (11), we note that c depends on the length of the line ℓ. The longer the line, thus, the faster the propagation of the electromechanical waves. This appears to be another inconsistency of the EWA as the travel speed is expected to depend on the properties of the medium (e.g., on the reactance per unit length) but not on its geometry (the length itself).

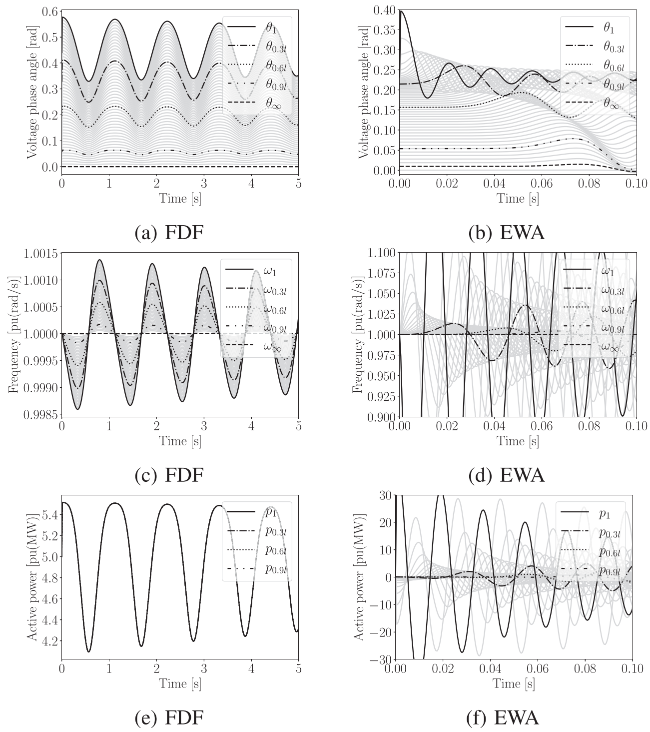


Fig. 1. OMIB system: FDF compared to EWA approach.

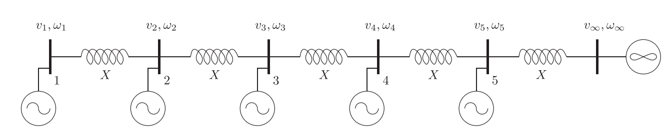


Fig. 2. 5-machine radial system.

V. Case Study

In this section we discuss two examples. The first example provides a comparison of the FDF-based continuum with the EWA, based on an one-machine infinite bus (OMIB) system. The second example is based on a multi-machine system and focuses on the ability of the FDF continuum to capture the evolution of electromechanical dynamics in a power network.

A. One-Machine Infinite-Bus System

Consider the example of a classical SM connected to an infinite bus through a lossless line of length l. The following parameter values are assumed: SM starting time $M = 20$ MWs/MVA; damping coefficient $D = 2$ pu; transient reactance $X'_{\mathrm{d}} = 0.3$ pu; line reactance $X = 0.085$ pu.

Fig. 1 shows the angle, frequency, and active power variations along the transmission line, as obtained using the FDF and the EWA. The disturbance considered is a step variation of the SM's rotor angle. Figs. 1(a) and 1(c) indicate that all points along the line respond instantaneously following the disturbance. On the contrary, Figs. 1(b) and 1(d) suggest a wave-like response, where points remote to the disturbance show a delayed reaction. Compared to the response of the FDF continuum which, as expected, lies in the electromechanical time scale, the EWA leads to an inaccurate representation of frequency variations, since it produces fast dynamics that are apparently spurious, see Fig. 1(b), (d), and (f). Small-signal stability analysis of the system further confirms these results. The deviations are a consequence of distributing the SM, i.e. of (9) as opposed to (4). Figs. 1(e) and (f) show that, as expected, the active power flow with the FDF is the same in every point of the line for any given t, whereas in the EWA, the active power varies as a continuum.

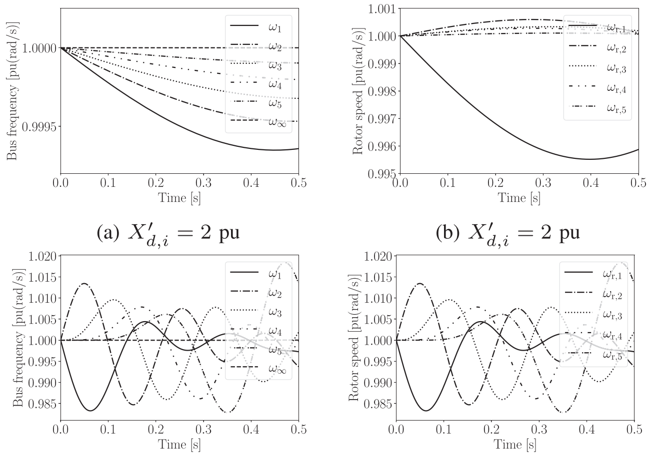
a) Multi-Machine Chain System

This section considers the 5-machine test system shown in Fig. 2. The following parameters are assumed: $M_i = 10$ MWs/MVA; $D_i = 3$ pu, $i = \{1, 2, \ldots, 5\}$; $X = 0.1$ pu. This system is not realistic *per se*. However, in the same vein as in [1] and following works based on the EWA, we assume that the system in Fig. 2 approximates a long power corridor of a complex and meshed grid.

The disturbance is a step variation of the rotor angle of the SM at bus 1. Fig. 3 shows the response of the FDF for two values of the SM transient reactances.

a) $X'_{\mathrm{d},i} = 2$ *pu:* Results show that, in this scenario, rotor speeds are not representative of the frequency variations at the buses to which the SMs are connected. Moreover, in the first instants after the disturbance, all buses appear to respond instantaneously, with the slope of the response at each bus depending on the distance from the disturbance location.

b) $X'_{\mathrm{d},i} \to 0$ *pu:* In this scenario, SM rotor speeds represent the bus frequency variations in the system. Some buses appear to respond with a considerable "delay." This effect is triggered



(a) $X'_{d,i} = 2$ pu (b) $X'_{d,i} = 2$ pu

(c) $X'_{d,i} \to 0$ pu (d) $X'_{d,i} \to 0$ pu

Fig. 3. 5-machine test system: FDF response.

by the values of the transient reactances being very small, which leads the initial slopes of the frequency variations at buses far away from the source of the disturbance to be close to zero. With the FDF and given that line dynamics are neglected, all buses respond instantaneously to the disturbance. Thus, potential bus responses that appear delayed are in effect simply slow. In chain-like systems that include multiple small machines, this time response is, ultimately, proportional to the electrical distance from the disturbance location.

VI. Conclusion

The letter shows that the linear distribution of the frequency along transmission lines can be deduced from the power flow equations under proper approximations. The letter also shows how this distribution leads to the FDF. The EWA, while imposing this linear distribution, ultimately leads to a wave model that may be inconsistent with the hypothesis on which it is built. The case study discusses two examples that provide a comparison between the FDF and the EWA. The comparison features limitations and inconsistencies of the EWA, and illustrates the ability of the FDF continuum to study the evolution of electromechanical dynamics, including frequency and power variations, in a power network. Future work will focus on removing some of the simplifications on which the FDF is based, and deduce the consequences of a more realistic model on the continuum modeling approach.

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